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Conceptual Knowledge of Fraction Arithmetic

Robert S. Siegler and Hugues Lortie-Forgues
Carnegie Mellon University and The Siegler Center for Innovative Learning, Beijing Normal University

Understanding an arithmetic operation implies, at minimum, knowing the direction of effects that the operation produces. However, many children and adults, even those who execute arithmetic procedures correctly, may lack this knowledge on some operations and types of numbers. To test this hypothesis, we presented pre-service teachers (Study 1), middle school students (Study 2), and math and science majors at a selective university (Study 3) with a novel direction of effects task with fractions. On this task, participants were asked to predict without calculating whether the answer to an inequality would be larger or smaller than the larger fraction in the problem (e.g., "True or false: 31/56 * 17/42 > 31/56"). Both pre-service teachers and middle school students correctly answered less often than chance on problems involving multiplication and division of fractions below 1, though they were consistently correct on all other types of problems. In contrast, the math and science students from the selective university were consistently correct on all items. Interestingly, the weak understanding of multiplication and division of fractions below 1 was present even among middle school students and preservice teachers who correctly executed the fraction arithmetic procedures and had highly accurate knowledge of the magnitudes of individual fractions, which ruled out several otherwise plausible interpretations of the findings. Theoretical and educational implications of the findings are discussed.

***Keywords**:* fractions arithmetic, conceptual knowledge, mathematical development, mathematical cognition, arithmetic

Mathematical knowledge during schooling predicts academic, occupational, and financial success years later. Even after controlling for other cognitive and demographic variables, mathematics achievement in high school is predictive of college matriculation, college graduation, and early career income (Murnane, Willett, & Levy, 1995). Especially striking, mathematics achievement at age 7 predicts socioeconomic status (SES) at age 42, even after statistically controlling for SES at birth, reading achievement, IQ, academic motivation, and years of education (Ritchie & Bates, 2013).

Among areas of mathematics, fractions (including decimals, percentages, ratios, rates, and proportions) seem to be especially important for later success. This central role is evident in fifth graders' fraction knowledge predicting their algebra knowledge and overall mathematics achievement in tenth grade, a relation that was present in both the U. K. and the U. S. even after controlling for IQ, reading comprehension, working memory, knowledge of whole number arithmetic, and parental education and income (Siegler et al., 2012). Fraction understanding also is essential for a wide range of occupations beyond STEM fields (science, technology, engineering and mathematics), including nurse, pharmacist, automotive technician, stone mason, and tool and die maker (Davidson, 2012; McCloskey, 2007; Sformo, 2008).

The importance of fractions makes it especially unfortunate that many children's, adolescents', and adults' fraction understanding is poor. On a recent National Assessment of Educational Progress (NAEP), a test presented to a large, nationally representative sample of U.S. students, only 50% of 8th graders correctly

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Robert S. Siegler and Hugues Lortie-Forgues, Department of Psychology, Carnegie Mellon University and The Siegler Center for Innovative Learning, Beijing Normal University.
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Correspondence concerning this article should be addressed to Robert S. Siegler or to Hugues Lortie-Forgues, Department of Psychology, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. E-mail: rs7k@andrew.cmu.edu or hugues1@andrew.cmu.edu

ordered from smallest to largest 2/7, 5/9, and 1/12 (Martin, Strutchens, & Elliott, 2007). Problems in understanding fraction magnitudes persist into adulthood; community college students choose the larger of two fractions on only about 70% of items, where chance is 50% correct (Schneider & Siegler, 2010; Stigler, Givvin, & Thompson, 2010). Teachers recognize the seriousness of the problem; a sample of 1,000 U.S. high school algebra teachers rated knowledge of fractions as one of the two largest weaknesses in their students' preparation for their course, from among 15 topics in mathematics (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Findings like these led the U. S. National Mathematics Advisory Panel (NMAP) to describe fractions as "the most important foundational skill not presently developed" (NMAP, 2008) and led the National Council of Teachers of Mathematics (2007) to emphasize the importance of improving teachers' and students' understanding of them. The problem is especially serious in the U.S., but it extends to countries that rank high on international comparisons of math knowledge, including Taiwan and Japan (Chan, Leu, & Chen, 2007; Yoshida & Sawano, 2002).

In addition to their importance for educational and occupational success, fractions are also crucial for theories of numerical development. As noted in the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011), learning fractions requires differentiating properties of natural numbers from properties of rational numbers. Students need to learn that each natural number has a unique successor but that infinitely many fractions fall between any two other fractions; that, multiplying two natural numbers never results in a product less than either factor (the numbers being multiplied), but that multiplying two proper fractions (fractions less than one) always does; and that dividing two natural numbers never results in a quotient greater than the number being divided, but dividing by a fraction less than one always does. Indeed, one the few major properties uniting natural numbers and fractions is that both express magnitudes that can be located and ordered on number lines.

A variety of types of data are consistent with this theoretical emphasis on numerical magnitudes. Accuracy of magnitude representations, as measured by performance on number line and magnitude comparison tasks, correlates consistently and quite strongly with overall mathematics achievement with both whole numbers (Geary, et al., 2007; Siegler & Booth, 2004) and fractions (Siegler & Pyke, 2013; Siegler, Thompson, & Schneider, 2011). Accuracy of numerical magnitude representations also predicts achievement test scores in later grades for both whole numbers (Geary, 2011; Watt, Duncan, Siegler, & Davis-Kean, 2014) and fractions (Bailey, Hoard, Nugent, & Geary, 2012; Jordan et al., 2013). Most important, magnitude knowledge is causally related to other aspects of mathematical knowledge. Randomized controlled trials aimed at improving knowledge of magnitudes produce gains not only in magnitude knowledge but also in arithmetic with both whole numbers (Booth & Siegler, 2008; Ramani & Siegler, 2008) and fractions (Fuchs et al., 2013; Fuchs et al., 2014).

The present study expands the integrated theory of numerical development beyond this emphasis on the central role of the magnitudes of individual numbers to include the role of understanding the magnitudes produced by arithmetic. The two issues show some striking similarities. For example, just as accuracy of estimation of individual fractions' magnitudes is related to mathematics achievement, so is estimation of the answers yielded by fraction addition (Hecht & Vagi, 2010). However, despite the parallel roles of magnitudes in the two areas, understanding individual fractions does not imply understanding of the magnitudes produced by fraction arithmetic..

Prior research has documented one type of misunderstanding of fraction arithmetic: the whole number bias. This well-documented error involves treating fraction numerators and denominators as independent whole numbers, as when claiming that 1/2 + 1/2 = 2/4 (Gelman, 1991; Hecht & Vagi, 2010; Mack, 2005; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). Although striking, this error occurs primarily among low-achieving students; for example, in a study that examined fraction arithmetic of middle school students, children whose achievement test scores were in the lowest one-third of the distribution made such errors on 32% of trials, whereas those whose achievement test scores were in the upper two-thirds made them on only 10% of trials (Siegler & Pyke, 2013). The frequency of such errors also decreases with age, falling from 25% among sixth graders to 16% among eighth graders in the same study.

In the present study, we examine a different type of misunderstanding of fraction arithmetic: *direction of effects errors*. At minimum, a person who understands an arithmetic operation should know the direction of effects that the operation produces. For addition and subtraction, the direction is the same for all positive numbers: addition produces answers greater than either addend, and subtraction produces answers less than the minuend (the number being subtracted from). However, for multiplication and division, the direction of effects depends on whether the numbers in the problem are greater or smaller than one. Multiplying numbers between zero and one always yields an answer less than either number being multiplied, and dividing by a number between zero and one always yields an answer greater than the number being divided (the dividend). Such knowledge has been studied with whole numbers (Baroody, 1992; Sophian & Vong, 1995) and has been claimed to be important with fractions, but to the best of our knowledge, it has not been studied systematically with fractions.

In principle, the directions of effects of arithmetic operations are easy to understand. If people think of fraction multiplication as "N of the M's," they should be able to predict the direction of effects, regardless of whether N and M are natural numbers (e.g., will 6 of the 4's be more than 6) or proper fractions (e.g., will 1/6 of the 1/4 be more than 1/4). Similarly, if people understand division as the number of times the divisor goes into the dividend, it should not be hard to realize that a divisor between 0 and 1 will go into the dividend more times than "1" would go into it, so the answer will exceed the dividend.

In contrast, if people understand multiplication as an operation that invariably increases the size of the numbers involved, and division as an operation that invariably decreases their size, as is the pattern with numbers greater than one, then they will perform below chance when both factors are proper fractions (fractions below one) or the divisor is.

Note that mastery of fraction arithmetic procedures – the algorithms or rules used to solve mathematical problems – could easily coexist alongside weak or inaccurate conceptual understanding of the procedures – implicit or explicit knowledge of how the procedures work, why they make sense, and how they are related to other procedures and concepts in the domain (Byrnes & Wasik, 1991; Hiebert & LeFevre, 1986; Rittle-Johnson & Schneider, in press; Rittle-Johnson & Star, 2007). Simply put, students might memorize fraction arithmetic procedures without understanding them. This is more than a theoretical possibility. Many U.S. teachers have weak conceptual understanding of fraction arithmetic (Ma, 1999; Lin, et al., 2013; Moseley, Okamoto, & Ishida, 2007; Rizvi & Lawson, 2007), which limits the understanding that they can convey to students. In principle, students who correctly execute fraction arithmetic procedures could induce some types of understanding by themselves. For example, they could infer that multiplying fractions below one results in answers less than either factor by connecting the magnitudes of the fractions being multiplied, the operation being performed, and the magnitudes of the answers. However, many students might not make such connections, either due to not attending to the magnitudes of fractions in the problems or due to forgetting them while executing the procedure. Thus, accurately executing fraction arithmetic procedures is no guarantee of understanding them.

The Present Study

Our main goal was to test the hypothesis that even many educated adults with years of whole number and fraction arithmetic experience nonetheless lack conceptual understanding of multiplication and division. This was hypothesized to be true even for people who execute fraction arithmetic procedures flawlessly and who accurately represent fraction magnitudes. Conceptual understanding of arithmetic operations was measured by a direction of effects task (e.g., Is $N_1/M_1 + N_2/M_2 > N_1/M_1$). This task, which in its present form is novel to this study, had two advantages for assessing conceptual understanding of arithmetic: It did not require highly accurate knowledge of the magnitudes of individual numbers (none for addition and subtraction; only whether each fraction exceeds one for multiplication and division), and application of the present analysis to it yielded eight predictions, one for each of the eight combinations of arithmetic operation and fraction magnitude (factors greater than or less than one).

Our main prediction was that frequency of accurate judgments would be below chance on multiplication and division problems with fractions less than one. This was predicted to be true even for people who flawlessly multiplied and divided fractions and who accurately estimated the magnitudes of individual fractions. In contrast, we also predicted that performance would be well above chance on addition and subtraction problems, regardless of the fractions involved, and on multiplication and division with fractions greater than one. The reason was that on these problems, either understanding the arithmetic operations or assuming that fraction arithmetic yields the same patterns as natural number arithmetic produces correct judgments.

These predictions were examined with three populations: pre-service teachers attending a school of education, students attending a middle school, and math and science majors at a highly selective university. The pre-service teachers attended a high quality college of education in Canada and provided a test of the main hypothesis with a sample of adults whose mathematical knowledge was above average for adults in North America. The middle school students were from a U. S. public school with typical achievement levels, according to performance on state tests. They were included to test whether the adults from Study 1 might have had conceptual understanding of the arithmetic operations when they studied them but forgotten that knowledge in the ensuing years. Finally, the math and science majors at the highly selective university were far above average in math achievement; including them tested whether the direction of effects task would lead even highly knowledgeable adults to perform poorly.

Study 1

Method

Participants. Participants were 41 pre-service teachers (4 men, 37 women, *Mean age* = 25.9 years, *SD* = 8.8) recruited from a French language public university in Montreal, Quebec, Canada. All were in a mathematical activities course in a program aimed at preparing them to teach kindergarten through sixth grade, a period in which fractions are a major focus of mathematics instruction. The teacher-training program was selective, with 53% of applicants being admitted. Self-reported R scores, a statistical index used in Quebec to measure incoming university students' academic performance, were provided by 26 of the 41 participants. Their scores were above the Quebec average of 25 (M = 28.87; SD = 2.63; versus M = 25, SD unknown) (CREPUQ, 2004). On the 2012 PISA, a test used to compare academic achievement of different countries and regions within countries, Quebec had the highest mean math achievement score of the 10 Canadian provinces; its mean PISA math score equalled that of Japan, and far exceeded that of the U.S. (CMEC, 2013).

Tasks. All measures were presented to the entire class in pencil and paper form.

Conceptual understanding of fraction arithmetic. This was measured by performance on the direction of effects task. On it, participants were asked to evaluate the accuracy of mathematical inequalities of the form a/b + c/d > a/b, where a/b was the larger of the fractions and the operation was either addition, subtraction, multiplication, or division. The fractions were chosen so that numerators and denominators were too large for the problem to be solved quickly via mental arithmetic. The fractions a/b and c/d were either both above one or both below one, and the operator was one of the four arithmetic operators, resulting in eight types of problems. Participants were presented 2 instances of each of the 8 types of problems, for a total of 16 items. The same pairs of operands -- 31/56 and 17/42, 41/66 and 19/35, 37/19 and 58/36, and 51/16 and 47/33 -- were presented for all four arithmetic operations.

Each page of the booklet included four problems, which included one instance of each arithmetic operation and one of each pair of operands. Participants were instructed not to compute the exact answer, but rather to decide without calculating whether the answer would be greater than the answer indicated in the inequality.

Fraction arithmetic computations. Each participant was presented four problems for each of the four arithmetic operations. For each operation, the operand 3/5 was combined with 1/5, 1/4, 2/3, and 4/5; for example, the subtraction problems were 3/5 - 1/5, 3/5 - 1/4, 2/3 - 3/5, and 4/5 - 3/5. Thus, all problems involved positive fractions less than one. Half of the problems had operands with equal denominators, and half with unequal ones. The larger operand was always on the left for subtraction and division problems.

Magnitudes of individual whole numbers and fractions. Knowledge of whole number magnitudes was measured by a 0-10,000 number line estimation task; knowledge of fraction magnitudes was measured by 0-1 and 0-5 number line tasks. On each trial, the number being estimated was printed above the midpoint of a 16 cm horizontal line with 0 just below the left end and 1, 5, or 10,000 just below the right end; the task was to mark where the target number belonged on the number line. On the whole number task, the target numbers were: 857, 1203, 2589, 3091, 4928, 5762, 6176, 7334, 8645 and 9410. On the 0-1 fractions task, they were: 1/19, 1/7, 1/4, 3/8, 1/2, 4/7, 2/3, 7/9, 5/6 and 12/13. On the 0-5 fractions task, they were: 1/19, 4/7, 7/5, 13/9, 8/3, 11/4, 10/3, 7/2, 17/4 and 9/2. On each magnitude estimation task, one target number was in each 1/10 of the line. Target fractions were also chosen to minimize correlations between the magnitude of the numerator and the magnitude of the whole fraction, so that basing placements solely on numerator magnitude would not yield accurate answers.

Procedure. All tasks were printed in a booklet and presented in a constant order: first the direction of effects task; then number line estimation with whole numbers, 0-1 fractions, and 0-5 fractions, in that order; and then fraction arithmetic. Two versions were generated by reversing the order of items within each task. The versions were randomly assigned to participants, who were tested in groups in their classroom.

Results and Discussion

Conceptual knowledge of fraction arithmetic. A repeated measures ANOVA with Fraction Size (above or below 1) and Arithmetic Operation (addition, subtraction, multiplication and division) as within-subject factors and number of correct judgments as the dependent variable yielded effects of Fraction Size $F(1, 40) = 23.17$, $p < .001$, $\eta_p^2 = 0.367$, and Arithmetic Operation $F(3, 38) = 36.26$, $p < .001$, $\eta_p^2 = 0.741$, both qualified by a Fraction Size x Arithmetic Operation interaction, $F(3, 38) = 6.24$, $p = .001$, $\eta_p^2 = 0.330$. As expected, post-hoc comparisons with the Bonferroni correction showed no differences between number of correct judgments with proper and improper fractions for addition (92% vs. 92%; $t(40) = 0$, $p = 1$; Hedges' $g = 0.00$) and subtraction (89% vs. 92%; $t(40) = 0.63$, $p = 0.53$; Hedges' $g = 0.00$). However, large differences between fractions below and above 1 were observed for multiplication (33% vs. 79%; $t(40) = 3.90$, $p < 0.001$; Hedges' $g = 1.11$) and division (29% vs. 77%; $t(40) = 4.11$, $p < 0.001$; Hedges' $g = 1.19$).

Analysis of individual response patterns yielded similar results. When considering all problems on the task, 42% of participants were correct on 100% of the 12 problems involving improper fractions and addition and subtraction of proper fractions, and on 0% on the 4 problems involving multiplication and division of proper fractions. Only 3% (1 participant) answered all 16 problems correctly.

Fraction arithmetic computations. A repeated measures ANOVA with Arithmetic Operation as a within subject variable and number correct as the dependent variable yielded an effect of Arithmetic Operation $F(3, 38) = 12.15$, $p < .001$, $\eta_p^2 = 0.490$. Post-hoc comparisons with a Bonferroni correction showed less accurate performance on fraction division ($M = 51\%$, $SD = 46\%$) than addition ($M = 87\%$, $SD = 25\%$; $t(40) = 4.76$, $p < 0.001$; Hedges' $g = 0.95$), subtraction ($M = 93\%$, $SD = 18\%$; $t(40) = 5.95$, $p < 0.001$; Hedges' $g = 1.17$) and multiplication ($M = 87\%$, $SD = 23\%$; $t(40) = 4.85$, $p < 0.001$; Hedges' $g = 0.97$).

The pre-service teachers rarely made the type of computational error predicted by the well-documented whole number bias (e.g., a/b + c/d = (a+b) / (c+d)). This error appeared on only 4.3% of addition and subtraction trials. Thus, the inaccurate judgments on the direction of effects task could not be attributed to the whole number bias.

Number line estimation. Performance on the number line tasks was measured by percent absolute error (PAE), computed as: (| Estimate - Correct answer |) / Numerical Range * 100. Thus, if a participant estimated the position of 1/4 to be 35% of the distance between 0 and 1, PAE on that trial was 10% ((35-25)/100). The higher the PAE, the less accurate the estimate. On the 3.3% of trials on which participants did not respond, PAE was computed using the value on the number line farthest from the correct answer (e.g., if the problem was to locate 1/19 on a 0-5 fraction number line, the estimate was computed as if it were at 5). This was done to yield a maximally conservative estimate of estimation accuracy. These trials occurred disproportionately among participants with weak mathematical knowledge and on relatively difficult problems, so deleting the trials would have distorted the data.

The most striking number line estimation finding was that the pre-service teachers had excellent knowledge of the magnitudes of fractions from 0-1, the same range in which they showed poor conceptual knowledge of multiplication and division. Estimates of fraction magnitudes in this range were as accurate as those for whole numbers in the 0-10,000 range, with both being highly accurate (For the 0-1 fraction task, *Mean PAE*= 4.41, *SD* = 3.07; for the 0-10,000 whole number task, *Mean PAE*= 3.53, *SD* = 2.64). Estimates of fractions on the 0-5 task were less accurate (*Mean PAE* = 14.94, *SD* =18.67).

Relations of performance across tasks. Relations between conceptual and procedural knowledge of arithmetic varied substantially on the four arithmetic operations (Table 1). On addition and subtraction, performance was strong on measures of both conceptual knowledge (the direction of effects task) and procedural knowledge (the arithmetic computation task); on multiplication, procedural knowledge was strong but conceptual knowledge was weak; on division, both were weak.

The discrepancy between conceptual and procedural knowledge of multiplication illustrated especially clearly that accurate execution of procedures does not imply understanding of their qualitative effects. The majority of participants (17 of 29) who correctly solved all four fraction multiplication problems erred on both of the direction of effects problems that assessed understanding of multiplication of proper fractions.

Although mean levels of performance on conceptual and procedural measures of division were similar, examination of individual participants' performance revealed a similar dissociation. Of participants who correctly answered all four of the fraction division problems, 47% (8 of 17) were incorrect on both of the direction of effects items that involved division of fractions below 1.

Table 1

Percent Correct on Assessments of Procedural and Conceptual Knowledge of Fraction Arithmetic: Pre-Service Teachers

Operation	Procedural knowledge	Conceptual knowledge
Addition	87	91
Subtraction	93	89
Multiplication	87	33
Division	51	30

Correlations among individual participants' performance on the direction of effects, arithmetic computation, and number line estimation tasks with fractions 0-1 were non-significant in 5 of 6 cases, revealing a similar dissociation. For multiplication, number of correct direction of effects judgments correlated *r* = .09 with number of correct arithmetic computations; number of correct direction of effects judgments correlated *r* = -.05 with number line PAE; and number line PAE correlated *r* = -.04 with number of correct arithmetic computations. For division, number of correct direction of effects judgments and arithmetic computations correlated *r* = .33; number of correct direction of effects judgments and number line PAE correlated *r* = -.11; and number of correct arithmetic computations and number line PAE correlated *r* = -.22.

In summary, pre-service teachers' conceptual understanding of fraction multiplication and division, as indicated by performance on the direction of effects task, was weak. This was true even for participants who flawlessly executed the fraction multiplication and division algorithms. Nor was the problem lack of fraction magnitude knowledge. Estimation of the magnitudes of fractions between 0 and 1, the range that elicited incorrect judgments of the qualitative effects of fraction multiplication and division, was highly accurate. Instead, the results indicated weakness in conceptual understanding of fraction arithmetic that was not attributable to weaknesses in knowledge of fraction arithmetic computation or magnitudes of individual fractions.

Study 2

In Study 2, we examined whether the Study 1 results replicated with a different population and age group: U.S. middle school students. This population allowed us to test whether students possess conceptual understanding of fraction arithmetic at the time they receive instruction in it. If so, the difference might reflect improvements in fraction instruction in the past decade, or it might reflect the pre-service teachers in Study 1 forgetting understanding they once had. The first seemed plausible, because mathematics instruction in general, and fraction instruction in particular, is focusing increasingly on inculcating conceptual understanding (NCTM, 2007). The latter interpretation also seemed plausible because once children learn correct procedures, they might forget the procedures' conceptual justifications.

Method

Participants. Participants were 59 6th and 8th graders (28 boys, 31 girls, *Mean age* = 12.9 years, *SD* = 1.19) from middle-income public schools near Pittsburgh.

Tasks. The same tasks were presented as in the previous study, plus one new task, which involved asking whether each fraction on the direction of effects task was greater or less than one. The purpose of this new task was to test whether failure to discriminate the effects of multiplying fractions above and below one might be due to inability to identify whether the fractions were above or below one. This task was presented after the others, so it could not affect performance on them.

Results and Discussion

Conceptual understanding of fraction arithmetic. As in Study 1, a repeated measures Fraction Size X Arithmetic Operation ANOVA on the direction of effects task yielded main effects of Fraction Size $F(1, 58) = 49.071$, $p < .001$, $\eta_p^2 = 0.458$, and Arithmetic Operation, $F(3, 56) = 28.351$, $p < .001$, $\eta_p^2 = 0.603$, both qualified by a Fraction Size x Arithmetic Operation interaction, $F(3, 56) = 19.07$, $p < .001$, $\eta_p^2 = 0.505$. Post-hoc comparisons with the Bonferroni correction again showed no differences between fractions below and above one for addition (89% vs. 92%; $t(58) = 1.36$, $p = 0.26$; Hedges' g =0.09) and subtraction (92% vs. 94%; $t(58) = 0.57$, $p = 0.57$; Hedges' $g = 0.07$), but large differences between them for multiplication (31% vs. 92%; $t(58) = 8.16$, $p < 0.001$; Hedges' $g = 1.74$) and division (47% vs. 70%; $t(58) = 2.24$, $p < 0.05$; Hedges' $g = 0.51$). Also as in Study 1, a fairly substantial number of participants (29%) were correct on 100% of the 12 problems involving fractions above one or addition and subtraction of fractions below one and on 0% of the 4 problems involving multiplication and division of fractions below one, and few participants (5 of 59) answered all problems correctly.

Fraction arithmetic computations. A repeated measure ANOVA with Arithmetic Operation as the sole variable yielded no significant effect. The children's performance on addition (78%), subtraction (83%), multiplication (81%) and division (82%) did not differ.

Number line estimation. Number line estimates were very accurate for whole numbers in the 0-10,000 range (*Mean PAE* = 4.08, *SD* = 1.67), fractions in the 0-1 range (*Mean PAE* = 3.57, *SD* = 2.05), and fractions in the 0-5 range (*Mean PAE* = 6.28, *SD* = 3.50).

These middle school students also were correct on 97% of judgments about whether the specific fractions on the direction of effects task were above or below 1. Thus, the poor performance on multiplication and division of fractions below one could not be explained by students lacking the magnitude knowledge required to perform well on it.

Relations of performance across tasks. As in Study 1, there was a clear dissociation between conceptual and procedural knowledge of fraction arithmetic (Table 2). Analysis of individual performance showed that among children who correctly solved all four of the fraction multiplication computation problems, 54% (19 of 35) erred on both direction of effects items for multiplication of fractions below 1. Similarly, among children who correctly solved all four fraction division computation problems, 39% (16 of 41) erred on both problems measuring understanding of division of proper fractions. Thus, with children as with adults, successful execution of fraction arithmetic computations was no guarantee of understanding the procedures.

Analyses of correlations among individual children's performance on the three tasks with fractions 0-1 yielded non-significant relations in all six cases. For multiplication, number of correct direction of effects judgments correlated $r = .16$ with number of correct arithmetic computations; number of correct direction of effects judgments correlated $r = -.10$ with number line PAE; and number line PAE correlated $r = -.20$ with number of correct arithmetic computations. For division, number of correct direction of effects judgments and arithmetic computations correlated $r = .13$; number of correct direction of effects judgments and number line PAE correlated $r = -.17$; and number of correct arithmetic computations and number line PAE correlated $r = -.13$.

Table 2

Percent Correct on Assessments of Procedural and Conceptual Knowledge of Fraction Arithmetic: Middle School Students

Operation	Procedural knowledge	Conceptual knowledge
Addition	78	89
Subtraction	83	92
Multiplication	81	31
Division	82	47

Study 3

An alternative interpretation to the results of Studies 1 and 2 was that the direction of effects task could not be understood, even by people who had strong mathematics understanding. To test this interpretation, we presented university students with very high mathematics achievement scores the same tasks and items as were presented in the previous two studies. The main prediction was that these students' deeper understanding of mathematics would allow them to consistently answer correctly all of the problems that examined conceptual understanding of fraction arithmetic.

Method

Participants. Participants were 17 undergraduate students (7 men, 10 women, *Mean age* = 19.9 years, *SD* = 1.3) majoring in computer science, engineering, physics, chemistry, or biology at a highly selective university. Self-reported SAT mathematics scores for the 16 of 17 students who reported them were in the 99th percentile (*M* = 778, *SD* = 23.73), a score consistent with mean SATs in these areas at the university.

Tasks and procedures. The tasks and procedures were identical to those in Studies 1 and 2, except that participants were tested individually or in pairs.

Results and Discussion

Conceptual knowledge of fraction arithmetic. Percent correct on the direction of effects task was very high on all eight types of problems (94% to 100% correct judgments, overall mean = 98%). A Fraction Size X Arithmetic Operation ANOVA, parallel to that in the first two studies, yielded no main effect or interaction. Most participants (77%) were correct on all 16 problems, none made more than two errors, and the few errors were unsystematically distributed across problems.

Table 3

Percent Correct on Assessments of Procedural and Conceptual Knowledge of Fraction Arithmetic: Math and Science Majors at a Selective University

Operation	Procedural knowledge	Conceptual knowledge
Addition	100	100
Subtraction	100	97
Multiplication	97	100
Division	99	97

Fraction arithmetic computations. Performance on all four operations was at ceiling, ranging from 97% correct for multiplication to 100% correct for addition and subtraction (overall mean = 99%). Thus, as shown in Table 3, both conceptual and procedural knowledge of arithmetic were uniformly excellent.

Number line estimation. These students' estimates were highly accurate on all three ranges: whole numbers from 0-10,000 (*Mean PAE* = 2.78, *SD* = 1.30), fractions from 0-1 (*Mean PAE* = 2.87, *SD* = 1.87), and fractions from 0-5 (*Mean PAE* = 4.17, *SD* =2.28).

Thus, results of Study 3 showed that students highly proficient in math exhibited strong conceptual understanding of fraction arithmetic. This ruled out the interpretation that the task precluded accurate judgments.

General Discussion

Middle school students and pre-service teachers demonstrated excellent understanding of the magnitudes of individual fractions between 0 and 1. However, the same middle school students and pre-service teachers demonstrated minimal understanding of the magnitudes produced by multiplication and division of fractions in the same range. They consistently predicted that multiplying two fractions below "1" would yield an answer greater than either factor and that dividing by a fraction below "1" would yield an answer smaller than the number being divided.

Additional results from the present study ruled out a variety of otherwise plausible explanations of these findings from the direction of effects task. The incorrect predictions on it were not due to the task being impossible; mathematics and science majors at a selective university performed almost perfectly on the same task. The incorrect predictions also were not due to the previously documented tendency to view fraction arithmetic in terms of independent combinations of the numerators and of the denominators. Participants showed such independent whole number errors on fewer than 10% of trials when solving fraction arithmetic problems (4.3% of addition and subtraction trials in Study 1 and 8.5% in Study 2), but erred on more than 70% of trials on the direction of effects task with multiplication and division of fractions below one. Inaccurate predictions on the direction of effects task also could not be attributed to lack of knowledge of individual fraction magnitudes or of how to execute fraction arithmetic procedures. The pre-service teachers in Study 1 generated extremely accurate estimates of the magnitudes of individual fractions and consistently solved fraction multiplication problems, yet they judged the direction of effects of fraction multiplication less accurately than chance. The results also suggested that inaccurate judgments on the direction of effects task could not be attributed to forgetting material taught years earlier. Sixth graders who had been taught fraction division in the same academic year and fraction multiplication one year earlier also performed below chance on the direction of effects task for these operations and numbers (30% correct for multiplication and 41% correct for division). These data from U.S. middle school students do not rule out the possibility that these Canadian pre-service teachers had forgotten relevant knowledge that they once had, but the data do make this possibility less likely.

One important unanswered question was whether the below chance performance was specific to common fractions or more general. To address this question, we conducted a pilot study on understanding of multiplication and division of decimals between 0 and 1. Ten arbitrarily chosen students from the same university as the pre-service teachers were presented the direction of effects task with decimals. Each common fraction from the direction of effects task was translated into its nearest 3-digit decimal equivalent (e.g., “31/56 * 17/42 > 31/56” became “0.554 * 0.405 > 0.554). The results with decimals closely paralleled those with common fractions: High accuracy (88% to 100% correct) on the six problem types in which fraction arithmetic produces the same pattern as natural number arithmetic, and below chance performance on the two problem types that showed the opposite pattern as natural numbers (38% correct on multiplication and 25% correct on division of fractions below one). Thus, the inaccurate judgments on the direction of effects task were general to multiplication and division of numbers from zero to one, rather than being limited to common fractions in that range.

Theoretical Implications

As noted in previous studies, understanding fractions requires recognizing that many properties of natural numbers are not properties of numbers in general (Siegler et al., 2011). The present study makes a related but different point: Understanding fraction arithmetic requires recognizing that many properties of natural number arithmetic are not properties of arithmetic in general. Although understanding individual fraction magnitudes and fraction arithmetic both require distinguishing properties of natural numbers from properties of all numbers, understanding the magnitudes of individual fractions does not imply understanding how fraction arithmetic operations transform those magnitudes.

The present findings also shed light on the sources of two types of common errors in fraction arithmetic (Siegler & Pyke, 2013). One involves inappropriately importing procedures from other fraction arithmetic operations into fraction multiplication. This leads to errors such as 3/5 * 4/5 = 12/5, in which the numerators of the two fractions are multiplied but the denominator is unchanged, as in addition and subtraction of fractions (e.g., 3/5 + 4/5 = 7/5). The other common error involves dividing fractions by inverting the numerator and multiplying (e.g., 2/3 ÷ 1/3 = 3/2 * 1/3 = 3/6 = 1/2) rather than inverting the denominator and multiplying (e.g., 2/3 ÷ 1/3 = 2/3 * 3/1 = 6/3 = 2). The present results suggest that weak understanding of the numerical magnitudes produced by fraction multiplication and division prevents people from rejecting these incorrect procedures on the basis that they yield implausible answers. Indeed, if learners believe that multiplication yields answers greater than either factor and that dividing yields answers smaller than the dividend, incorrect answers will often seem *more plausible* than correct ones. A student who believes that multiplication should always yield answers larger than either factor might well view, “3/5 * 4/5 = 12/5,” as more plausible than the correct answer, “3/5 * 4/5 = 12/25.”

The present findings narrowed the range of potential explanations of weak conceptual understanding of fraction multiplication and division, but left open at least two interpretations of the underlying difficulty. One possibility is that many children and adults have a specific belief that arithmetic operations produce the same direction of effects regardless of the numbers involved. In particular, they might believe that for all numbers, addition and multiplication yield answers greater than either operand, and subtraction and division yield answers smaller than the larger operand. Such a belief would reflect a conclusion based on the pattern of answers yielded by natural number arithmetic. Each natural number operation has the same direction of effects regardless of the numbers in the problem (excepting multiplication and division by 1, which are often explicitly described as "exceptions"). Illustrative of this logic, when asked to "Try to explain what 1/2 ÷ 1/4 means," a sixth grader in a pilot study wrote "You're making 1/2 1/4 times smaller than it was"; when asked to explain what 1/2 * 1/4 means, the child wrote "You are making 1/2 1/4 times bigger."

An alternative interpretation is that many people have no strong beliefs about the results yielded by multiplication and division of fractions, and therefore rely on the pattern with natural numbers as a default option. Within this interpretation, which resembles one proposed for some pre-service teachers by Simon et al. (2010), participants applied their understanding of natural number arithmetic to the direction of effects task with fractions not because they were convinced that this was correct but because they did not know what else to do. Relying on default knowledge may be quite common in situations where people lack understanding but need to do something. For example, if asked how a catalytic converter works, many people might rely on knowledge of how air conditioning filters work, not because they are convinced that catalytic converters work that way, but due to inability to generate a better explanation (see Rozenblit & Keil, 2002, for extensive documentation of such default explanations). Consistent with this interpretation, Ma (1999) found that most U.S. teachers in her study could not generate any explanation of what 1 3/4 ÷ 1/2 means, or resorted to explaining a different problem (1 3/4 ÷ 2). Similarly, an eighth grader in the pilot study mentioned in the previous paragraph explained 1/2 * 1/4 by writing "It means taking half a pie, 1/4 of another pie, and showing how much you'd get" and could not generate any explanation of the meaning of 1/2 ÷ 1/4. Obtaining confidence ratings regarding direction of effects judgments could help discriminate between these possibilities.

Another unresolved issue is whether the weak understanding is specific to fraction multiplication and division or whether it extends to all multiplication and division. Consistent with the view that the difficulty is specific to fractions, children often provide reasonable explanations of natural number multiplication and division. They explain natural number multiplication in terms of repeated addition (Lemaire & Siegler, 1995) and natural number division in terms of either multiplication (72 ÷ 8 = 9 because 8 * 9 = 72) or repeated addition (72 ÷ 8 = 9 because adding 9 8 times = 72) (Robinson, et al., 2006). However, teachers often explain multiplication and division in these ways, so children might just be repeating what they have been told and might not have as deep an understanding of multiplication and division as these explanations suggest (Dubé & Robinson, 2010; Robinson & Dubé, 2009). Investigating in greater depth people's understanding of multiplication and division with natural numbers as well as fractions, and analyzing how the two are related, seems a promising route for future research.

Educational Implications

Results of Studies 1 and 2 indicated that even people with very accurate knowledge of fraction arithmetic procedures and magnitudes of individual fractions often possess weak conceptual understanding of multiplication and division of fractions below one. These findings are consistent with previous results from mathematics education research (e.g., Heller, et al., 1990; Lamon, 2007) as well as the recommendations of mathematics education organizations (e.g., NCTM, 1989), and point to the need to explicitly teach qualitative reasoning about fraction arithmetic (Behr, et al., 1992; Huinker, 2002).

A related, straightforward instructional implication is that teachers and textbooks should emphasize that multiplication and division produce different outcomes depending on whether the numbers involved are greater than or less than 1, and should discuss why this is true. Chinese textbooks include such instruction. For instance, in a lesson on multiplication with decimals, Chinese students are asked to solve and discuss answers to the following three problems: 4.9 * 1.01; 4.9 * 1; 4.9 * 0.99 (Sun & Wang, 2005). The same could be done with triads of fractions, such as 1/2 * 8/7, 1/2 * 7/7, and 1/2 * 6/7. Such well-chosen problems and accompanying discussion are likely to create meaningful and memorable knowledge of the effects of multiplying by numbers above and below one.

Another instructional implication is that teachers and textbooks should focus more attention on the shifting meaning of wholes in the context of fraction arithmetic. For example, when asked to use multiplication to find what 1/2 of 1/4 of a pie is, the relevant whole for 1/4 is the whole pie, but the relevant whole for 1/2 is the 1/4 of the pie. Several investigators have noted that many students fail to gain such understanding of wholes in the context of fraction arithmetic and have suggested means for remedying the difficulty (e.g., Ball, 1990; Simon, 1993; Tobias, 2013). Instruction that successfully inculcatesPh such understanding would provide a strong foundation for efforts to teach children what fraction multiplication means.

A further instructional implication is that instruction should emphasize understanding of commonalities uniting the processes of whole number and fraction arithmetic. One approach to attaining this goal is to adopt linguistic phrasings of multiplication and division that promote recognition that these operations have the same meaning with whole and with rational numbers. For instance, multiplication of two fractions (e.g., 1/3 * 1/5) can be expressed as "How much is 1/3 of the 1/5"; multiplication of a whole number and a fraction as “How much is 1/5 of the 3”; and multiplication of two whole numbers as “How much is 5 of the 3’s.” Using this phrasing of multiplication first with two whole numbers, then with a whole number and a fraction, and then with two fractions might be especially effective for promoting analogies from better to less understood cases, an instructional strategy that has proved effective in many domains (Gentner & Holyoak, 1997). Illustrating the multiplicative process in all cases in the common format of a number line might deepen understanding of multiplication further, by providing spatial as well as verbal support for a shared representation. Discussing with students why multiplying two numbers below one always results in an answer less than one, why multiplying two numbers above one always results in an answer greater than one, and why multiplying a number less than one by a number greater than one sometimes results in one outcome and sometimes in the other could deepen understanding further.

A similar, though somewhat more complex, approach could be used with division. Students could be encouraged to first judge whether the problem involves dividing a larger by a smaller number or dividing a smaller by a larger number. If the problem involves division of a larger by a smaller number, for example, 5/8 ÷ 1/8, students could be encouraged to think of the problem as “How many times can N go into M” (e.g., “How many times can 1/8 go into 5/8”). If the problem involves division of a smaller by a larger number, such as 1/8 ÷ 5/8, students could be encouraged to think of the problem as “How much of N can go into M” (e.g., “How much of 5/8 can go into 1/8”). As in multiplication, these phrasings can be applied to whole numbers as well as fractions (“How many times can 8 go into 48” “How much of 48 can go into 8”), and it seems likely to prove useful to proceed from using each phrasing first with two whole numbers, then with a whole number and a fraction, and then with two fractions (maintaining whether the problems involve a larger number divided by a smaller number or the opposite). Also as in multiplication, these phrasings lend themselves to a common spatial representation of the workings of the operation on a number line.

We believe that these alternate phrasings provide a clearer indication of what multiplication and division mean than the usual phrasings. Their applicability to both natural numbers and fractions seems a promising way of helping students understand that the mathematical operations have the same meaning in both cases, though their effects on numerical magnitudes vary with the numbers involved. We plan to test the efficacy of these instructional ideas in the near future.

Limitations and Future Research Directions

The present study has several limitations, each of which suggests directions for future research. Perhaps the most important limitation is that the present study examined only one aspect of understanding of fraction arithmetic – the direction of effects of fraction arithmetic operations. Future research should also examine other aspects of conceptual understanding of fraction arithmetic, including the approximate magnitudes produced by fraction arithmetic operations, the correspondences that can be drawn between fraction arithmetic operations and models of their effects, and the varying wholes that correspond to the operands in fraction arithmetic problems.

A second limitation is that the study did not relate the fraction instruction that students had received to their conceptual and procedural knowledge of fraction arithmetic. Of particular interest for future research is determining whether children who receive conceptually oriented fraction instruction subsequently show superior conceptual understanding of fraction arithmetic and also superior knowledge and memory of the fraction arithmetic procedures.

A third limitation of the present research is that the three samples in the three studies came from different schools in different countries, and might well have received different fraction instruction. This left open the possibility that the Canadian adults might previously have better understood fraction multiplication and division, even though neither they nor the U.S. 6th and 8th graders, who had recently received relevant instruction, showed such conceptual understanding. A longitudinal study tracking the same people over time would be ideal for determining if, and to what extent, such loss of earlier conceptual knowledge occurred. The present findings that conceptual understanding of fraction multiplication and division is weak even among children and adults who have excellent knowledge of fraction magnitudes and arithmetic procedures, suggest that addressing these issues is well worthwhile.

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Correction to Siegler and Lortie-Forgues (2015)

In the article "Conceptual Knowledge of Fraction Arithmetic" by Robert S. Siegler and Hugues Lortie-Forgues (*Journal of Educational Psychology,* Advance online publication, January 19, 2015. http://dx.doi.org/10.1037/edu0000025), there were two rounding errors in Table 1. The value 91 in the upper-right corner should be changed to 92, and the value 30 in the bottom-right corner should be changed to 29. Table 1, 2, and 3 also refer only to fractions between 0 and 1, and not fractions above one.

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