Chapter 20

Implications of Cognitive Science Research for Mathematics Education

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Over the past 20 years, a great deal of cognitive science research has focused on mathematics learning. The majority of this research has examined basic capabilities such as counting, understanding of numerical magnitudes, arithmetic (both word problems and purely numerical problems), and pre-algebra. Another, somewhat smaller, body of research has been devoted to students' understanding of, and learning about, algebra, geometry, and computer programming. This research now allows relatively firm conclusions to be drawn about a number of aspects of mathematics learning relevant to the NCTM standards. This paper focuses on 8 areas in which such conclusions can be drawn:

1. Mathematical understanding before children enter school
2. Pitfalls in mathematics learning
3. Cognitive variability and strategy choice
4. Individual differences
5. Discovery and insight
6, Relations between conceptual and procedural knowledge
7. Cooperative learning
8. Promoting analytic thinking and transfer

The remainder of this paper examines conclusions based on cognitive science research on each of these topics. This work informs us regarding how children typically learn particular skills and concepts, the stumbling blocks that many of them encounter, and instructional practices that can produce greater learning. The examples that are discussed focus on acquisition of particular mathematical procedures and concepts, rather than broad philosophical issues about the nature of children as learners or about what mathematics should be taught. These latter issues are largely outside the purview of cognitive science approaches, or indeed any empirically based approach to mathematics learning. Further, one of the main lessons of cognitive science research is that successful teaching and learning depends on careful, detailed, analysis of the particulars of individual children learning particular skills and concepts. Therefore, this chapter deliberately steers clear of statements about math learning in general, instead focusing on findings concerning how children learn certain key ideas and procedures.

Mathematical Understanding Before Children Enter School

Children’s learning of mathematics in school contexts builds on a substantial base of understandings that they acquire before they begin their formal education. Understanding what children already know when they enter school is critical both for identifying what they still need to be taught and also for identifying strengths on which further instruction can be based. Some of these acquisitions are universal; others depend on environments within which children develop. Differences among social and cultural groups in the degree to which they master this latter, more variable, group of skills is clearly related to subsequent differences in learning of mathematics at school. Also striking is the fact that from early in life, children possess relatively abstract, as well as concrete, understanding of numbers and mathematical concepts.

Cardinality

One fundamental underpinning of understanding of mathematics, the concept of cardinality, appears to be universally present from the first months out of the womb. By four months of age, perhaps earlier, infants can discriminate one object from two, and two objects from three (Antell & Keating, 1983; Starkey, Spelke & Gelman, 1990; van Loosbroek & Smitsman, 1990). This was learned through the use of the habituation paradigm. Infants were shown a sequence of pictures, each of which contained a small set of objects, such as three circles. The sets differed from trial to trial in size of the objects, brightness, distance apart, and other properties, but they always had the same number of objects. Once the infants habituated to displays with this number of objects, they were shown a set that was comparable in other ways to the displays they had seen but that had a different number of objects. Their renewed looking attested to their having abstracted the number of objects in the previous sets.

These nascent understandings of cardinality also make it possible for infants to realize the consequences of adding and subtracting small numbers of objects (Simon, Hespos, & Rochat, 1995; Wynn, 1992). In one task, 5-month-olds saw one or two objects, then saw a screen come down in front of them, then saw a hand place another object behind the screen, and then saw the screen rise. Sometimes the result was what would be expected by adding the one new object to the one or two that were already behind the screen; other times (through trickery) it was not. The infants looked for a longer time when the number of objects was not what it should have been, thus suggesting that they expected the correct number of objects to be present.

Not until three or four years later, however, do children discriminate among even slightly larger numbers of objects, such as four objects versus five or six (Starkey & Cooper, 1980; Strauss & Curtis, 1984) This suggests that the competencies that develop in infancy are produced by *subitizing,* a quick and effortless process of recognition that people can apply only to sets of one to three or four objects. When we see a row of between one and four objects, we feel like we immediately know how many there are; in contrast, with larger numbers of objects, we usually feel less sure and often need to count. Adults and 5-year-olds are similar to infants in being able to very rapidly identify the cardinal value of one to three or four objects, but not larger sets, through subitizing (Chi & Klahr, 1975).

At 3 or 4 years of age, children become proficient in another means of establishing the cardinal value of a set--*counting*. This allows them to assign numbers to larger sets than can be subitized. Gelman and Gallistel (1978) noted the rapidity with which children learn to count and identified a set of *counting principles* on which this rapid learning seems to be based. The fact that preschoolers possess such principles is particularly important because it indicates that from early in learning, children's understanding of mathematics includes abstract knowledge, as well as set procedures and factual information. Equally important, from the beginning, the abstract knowledge influences learning and execution of procedures. The five counting principles that Gelman and Gallistel identified were:

1. The *one-one principle*: Assign one and only one number word to each object.
2. The *stable order principle*: Always assign the numbers in the same order.
3. The *cardinal principle*: The last count indicates the number of objects in the set.
4. The *order irrelevance principle*: The order in which objects are counted is irrelevant.
5. The *abstraction principle*: The other principles apply to any set of objects.

Several types of evidence indicate that children understand all of these principles by age 5, and some of them by age 3 (Gelman & Gallistel, 1978). Even when children err in their counting, they show knowledge of the one-one principle, since they assign exactly one number word to most of the objects. For instance, they might count all but one object once, either skipping or counting twice the single miscounted object. These errors seem to be ones of execution rather than of misguided intent. Children demonstrate knowledge of the stable order principle by almost always saying the number words in a constant order. Usually this is the conventional order, but occasionally it is an idiosyncratic order such as "1, 3, 6." The important phenomenon is that even when children use an idiosyncratic order, they use the same idiosyncratic order on each count. Preschoolers demonstrate knowledge of the cardinal principle by saying the last number with special emphasis. They show understanding of the abstraction principle by not hesitating to count sets that include different types of objects. Finally, the order irrelevance principle seems to be the most difficult, but even here, 5-year-olds demonstrate understanding. Many of them recognize that counting can start in the middle of a row of objects, as long as each object is eventually counted. Although few children can state the principles, their counting suggests that they know them.

Ordinality

Mastery of ordinal properties of numbers, like mastery of cardinal properties, begins in infancy. However, it seems to begin a little later, between 12 and 18 months.

The most basic ordinal concepts are *more* and *less*. To test when infants understand these concepts as they apply to numbers, Strauss and Curtis (1984) found that 16- to 18-month-olds have a rudimentary understanding of these concepts. Babies who had been reinforced for reaching for a square with 2 dots rather than 1, and then one with 3 dots rather than 2, subsequently selected a square with 4 dots rather than 3, thus indicating understanding of the ordinal property "more numerous". They also succeeded when the square with fewer dots was reinforced.

As with cardinality, extension of these early understandings of ordinality beyond sets with a few objects takes a number of years. The task most often used to examine later understandings of ordinality involve asking such questions as "Which is more: 6 oranges or 4 oranges." Not until age 4 or 5 years can children from middle class backgrounds solve these ordinality problems consistently correctly for the numbers from 1 to 9 (Siegler & Robinson, 1982). The greatest difficulty occurs with numbers that are relatively large and close together (e. g., 7 vs 8). Counting skills may be important in development of this ordinal knowledge as well as in arithmetic; the number that occurs later in the counting string is always the larger number, and it is easier to remember which number comes later when the numbers are far apart in the counting string.

Although most middle income children know the relative magnitudes of all of the single-digit numbers when they enter school, comparable understanding does not exist among children from low-income backgrounds, at least within the United States (Griffin, Case, & Siegler, 1994). Such children often have little or no sense of the relative magnitudes of single-digit numbers when they enter first grade. This lack of understanding makes it particularly difficult for them to understand the basis of simple arithmetic operations, which is likely related to their slow learning of the basic arithmetic facts (Jordan, Huttenlocher, & Levine, 1995). Relatively poor counting skills also appear to contribute to these children’s difficulty in learning both numerical magnitudes and arithmetic.

A similar point is relevant for interpreting differences in standardized achievement test scores between children in the U.S. and in other countries. The relatively poor performance on these international comparisons is often attributed to formal instruction being inferior in the U.S. However, substantial differences between arithmetic knowledge of children in the U.S. and East Asia exist before children in either country receive formal instruction in arithmetic (Geary, et al., 1993). This does not mean that mathematics education in U.S. schools is as effective as in East Asian schools, but it does demonstrate that differences in math achievement in different countries reflects cultural differences that influence math learning outside the classroom as well as inside it.

Pitfalls in Mathematics Learning

As noted in the last section, from the preschool years onward, children learn abstract mathematical concepts and principles, as well as procedures and facts. Fairly often, however, they either fail to grasp the concepts and principles that underlie procedures or they grasp relevant concepts and principles but cannot connect them to the procedures. Either way, children who lack such understanding frequently generate flawed procedures that generate systematic patterns of errors. Depending on how one looks at it, these systematic errors can be seen as either a problem or an opportunity. They are a problem in that they indicate that children do not know what we have tried to teach them. On the other hand, they are an opportunity, in that their systematic quality points to the source of the problem, and thus indicates the specific misunderstanding that needs to be overcome. Examples can be found in many areas of math learning. Here, I will examine three areas with particularly prominent systematic misconceptions: the long subtraction algorithm, arithmetic and magnitude comparison involving fractions, and algebraic equations.

Buggy Subtraction Algorithms

Brown and Burton (1978) investigated acquisition of the multidigit subtraction algorithm. They used an error analysis method which involved first presenting problems on which incorrect rules ("bugs") would lead to specific errors, and then examining individual children's pattern of correct answers and errors to see if they fit the pattern that would be produced by a buggy rule.

Many of children's errors reflected such bugs. Consider the pattern in Table 1. At first glance, it is difficult to draw any conclusion about this boy's performance, except that he is not very good at subtraction. With closer analysis, however, his performance becomes understandable. All three of his errors arose on problems where the *minuend* (the top number) included a zero. This suggests that his difficulty was due to not understanding how to subtract from zero.

307	856	606	308	835
-182	-699	-568	-287	-217
285	157	168	181	618

Fig. 20.1. Example of a subtraction bug

Analysis of the problems on which the boy erred (the first, third, and fourth problems from the left) and the answers that he advanced suggests the existence of two bugs that would produce these particular answers. Whenever a problem required subtraction from 0, he simply flipped the two numbers in the column with the 0. For example, in the problem 307 - 182, he treated 0 - 8 as 8 - 0, and wrote "8" as the answer. The boy's second bug involved not decrementing the number to the left of the zero (not reducing the 3 to 2 in 307 - 182). This lack of decrementing is not surprising because, as indicated in the first bug, the boy did not borrow anything from this column. Thus, the three wrong answers, as well as the two right ones, can be explained by assuming a basically-correct subtraction procedure with two particular bugs.

Although such bugs are common among American children, they are far less common among Koreans (Fuson & Kwon, 1992). A major reason appears to be that Korean children have a firmer grasp of the base-10 system and its relation to borrowing. Such understanding makes it more likely that children's borrowing will maintain the value of the original number.

Fractions

When presented the problem 1/2 + 1/3, many children answer 2/5. They generate such answers by adding the two numerators to form the sum's numerator and by adding the two denominators to form its denominator. The misunderstanding is far from transitory. Many adults enrolled in community college math courses make the same mistake (Silver, 1983).

Much of children's difficulty in fractional arithmetic arises from their not thinking of the magnitude represented by each fraction. This is evident in children's errors in estimating the answer to 12/13 + 7/8 (Table 2). On a national achievement test, fewer than one-third of U. S. 13- and 17-year-olds accurately estimated the answer to this simple problem (Carpenter, et al., 1981). Yet how could adding two numbers that were each close to 1 result in a sum of 1, 19, or 21?

Estimate the answer to 12/13 + 7/8. You will not have time to solve the problem using paper and pencil.

	Percentage Choosing Answer	
Answer	Age 13	Age 17
1	7	8
2	24	37
19	28	21
21	27	15
I don't know	14	16

Fig. 20.2. Estimating the sum of two fractions (Carpenter, et al., 1981, p. 36).

A similar misunderstanding of the relation of symbols to magnitudes is evident in children's attempts to deal with decimal fractions. Consider how they judge the relative size of two numbers, such as 2.86 and 2.357. The most common approach of fourth and fifth graders on such problems is to say that the larger number is the one with more digits to the right of the decimal point (Resnick, et al., 1989). Thus, they would judge 2.357 larger than 2.86. Such choices appear to be based on an analogy between decimal fractions and whole numbers. Since a whole number with more digits is always larger than one with fewer digits, some children assume that the same is true of decimal fractions.

Another group of children made the opposite responses. They consistently judged that the larger number was the one that had fewer digits to the right of the decimal. Thus, 2.43 would be larger than 2.897. Many of these children reasoned that .897 involves thousandths, .43 involves hundredths, hundredths are bigger than thousandths, so .43 must be bigger than .897.

The difficulty in understanding decimal fractions does not quickly disappear. Zuker (1985; cited in Resnick et al., 1989) found that one-third of seventh and ninth graders continued to make one of the two errors described above. Thus, with decimal fractions as with long subtraction, children's failure to understand the number system, or to link that understanding to specific procedures, leads to systematic and persistent errors.

Algebraic Equations

Systematic errors also are evident in children's efforts to represent concrete situations in algebraic equations. Even students who do well in algebra classes often do so by treating the equations as exercises in symbol manipulation, without any connection to real-world contexts.

This superficial understanding creates a situation in which mistakes often arise. Many such mistakes arise from incorrect extensions of correct rules (Matz, 1982; Sleeman, 1985). For example, since the distributive principle indicates that

$$a \times (b + c) = (a \times b) + (a \times c),$$

some students draw superficially-similar conclusions, such as

$$a + (b \times c) = (a + b) \times (a + c).$$

Students use a variety of procedures to determine whether transformations of algebraic equations are appropriate. Among 11- to 14-year-olds, the most frequent strategy is to insert numbers into the original and transformed equations to see if they yield the same result (Resnick, Cauzinille-Marmeche, & Mathieu, 1987). This procedure reveals whether the transformation is allowable, though it rarely indicates why. Another common approach is to justify the transformation by citing a rule. Some students cite appropriate rules, but many others cite distorted versions of rules, such as the incorrect version of the distributive law cited above.

These problems are not quickly overcome. Even college students encounter difficulty with them. For example,more than one-third of freshman engineering students at a major state university could not write the correct equation to represent the simple statement "There are six times as many students as professors at this university" (Clement, 1982). Most wrote 6S=P, which reflects a superficially reasonable but deeply flawed understanding of the relation between algebratic equations and the situations theyrepresent. On a more positive note, such errors also indicate the source of the problem--in this case, that students are not analyzing in any depth the relation between what they have written and the problem they are representing. Deliberate practice in representing problem situations in equations, and analyzing why various equations do or do not accurately represent the problem situation, seems likely to be helpful in overcoming this problem.

More generally, research on children's systematic errors points to a central lesson that has emerged from cognitive science research: The importance of cognitive task analysis. To promote effective learning, it is essential to analyze in detail the particular procedures and concepts to be learned (Anderson, Reder, & Simon, 1997), to provide students with instruction and examples that help them learn the component skills and understandings, to anticipate types of misunderstandings that most often arise in the learning process, and to be prepared with means for helping students move beyond these misunderstandings.

Cognitive Variability and Strategy Choice

Children's thinking has often been described as something like a staircase, in which children first use one approach to solve problems, then adopt a more advanced approach, and later adopt a yet more advanced approach. For example, students of children's basic arithmetic (e.g., Ashcraft, 1987) have proposed that when children start school, they add by counting from one; sometime during first grade, they switch to adding by counting from the larger addend; and by third or fourth grade, they add by retrieving the answers to problems.

More recent studies, however, have shown that children's thinking is far more variable than such staircase models suggest. Rather than adding by using the same strategy all of the time, children use a variety of strategies from early in learning, and continue to use both less and more advanced approaches for periods of many years. Thus, even early in first grade, the same child, given the same problem, will sometimes count from one, sometimes, count from the larger addend, and sometimes retrieve the answer. Even when children master strategies that are both faster and more accurate, they continue to use older strategies that are slower and less accurate as well. This is true not just with young children, but with preadolescents, adolescents, and even adults (Kuhn, Garcia-Mila, Zohar, & Anderson, 1995; Schauble, 1996).

This cognitive variability is a spontaneous feature of children's thinking. Efforts to change it do not usually meet with much success. For example, in one study, first-to-third-grade teachers were interviewed regarding their beliefs about their students' arithmetic strategies and their evaluation of whether the students' use of multiple strategies was a good thing (Siegler, 1984). All of the teachers recognized that the children used multiple strategies, though most viewed this as a bad thing. One teacher said that she was constantly discouraging students from using strategies such as counting on their fingers. When asked how often she had done this with the pupil in her class who did it the most often, she asked, "How many days have there been in the school year so far?" This teacher and others recognized that even when they explicitly told students not to use their fingers, they did anyway, even if they had to do it by putting their fingers in their lap, under their leg, or behind their back.

There is a certain logic that supports the teacher's view. Older students and those who are better at math don't use their fingers, whereas younger and less-apt students do. One goal of education is to make younger and less apt students more like older and more apt ones. Therefore, children who use their fingers should be discouraged from doing so.

However, children actually learn better when they are allowed to choose the strategy that they wish to use. Immature strategies generally drop out naturally when students have enough knowledge to answer accurately without them. Even basic strategies such as counting fingers allow students to generate correct answers when forbidding use of the strategies would lead to many errors. Further, students who use a greater variety of different strategies for solving problems also tend to learn better subsequently (Alibali & Goldin-Meadow, 1993; Chi et al., 1994; Siegler, 1995). This is in part because the greater variety leads the students to be able to cope with whatever kinds of problems they encounter, rather than just being able to cope with a narrow range. Allowing children to use the varied strategies that they generate, and helping them understand why superficially-different strategies converge on the right answer and why superficially reasonable strategies are incorrect seems likely to build deeper understanding (Siegler, forthcoming).

Children's use of diverse strategies makes it essential that they choose appropriately among the strategies. To choose appropriately, they must adjust both to situational variables and to differences among problems. Situational variables include time limits, instructions, and the importance of the task. For example, in a magic-minute exercise, it is adaptive for children to state answers quickly, even if they aren't absolutely sure of them. Similarly, if it's very important to be correct in the particular situation, then checking the correctness of answers becomes more worthwhile. At least from second grade onward, children shift their choices appropriately to adapt to such situational variations.

Adaptive choice also involves adjusting strategy use to the characteristics of particular problems. When children are faced with a simple problem, it often is ideal for them to use a strategy that can be executed quickly, because it will be sufficient to solve the problem. In contrast, when faced with a more difficult problem, they may need to adopt a more time consuming and effortful strategy to generate the correct answer. Adaptive choice involves using quick and easy strategies when they are sufficient, and using increasingly effortful ones when they are necessary to be correct.

Research on strategy choice has also revealed some surprising similarities in the performance of children from different socio-economic groups. Children from low-income backgrounds, particularly low-income African-American backgrounds, are often depicted as choosing strategies unwisely. Suggestions have been made that instruction focus on improving their metacognition and their strategy selection. However, selection of appropriate strategies does not seem to be their main problem, at least in the context of arithmetic. Their strategy choices are just as systematic and just as sensitive to problem characteristics as those of children from middle income backgrounds (Kerkman & Siegler, 1993). Instead, their problem seems to be that they do not possess adequate factual knowledge. This in turn seems to be due to less practice in solving problems, and to less good execution of strategies, rather than to any high level deficiency in their thinking. The findings indicate that greater practice and instruction in how to execute strategies may be the most useful approach to improving their arithmetic skills.

Individual Differences

Substantial individual differences exist in cognitive variability and in the kinds of strategy choices that children make. These involve both differences in knowledge and differences in cognitive style. As early as first grade, children can be divided into three groups on the basis of their strategy choices in arithmetic: good students, not-so-good students, and perfectionists (Siegler, 1988). Good students and not-so-good students differ in all the ways that would be expected from the names. The good students are faster, more accurate, use more advanced strategies, and perform better on standardized math achievement tests.

The differences between the perfectionists and the other two groups are more interesting. The perfectionists are just as accurate as the good students. They also have equally high math achievement and equally high IQ scores. Their performance in later grades also appears to be equally strong (Kerkman & Siegler, 1993). However, in terms of their strategy choices, they choose a higher proportion of slow and effortful strategies. Unless they are very sure of the answer, they don't rely on memory, preferring instead to use such strategies as counting from one or from the larger addend. This appears to reflect a stylistic preference. Perfectionists seem to set a very high criterion for when they are sure enough to state an answer without checking it via a backup strategy, such as counting on their fingers. They state retrieved answers on the easiest problems, but only on them, whereas good students, with comparable knowledge, state retrieved answers on a considerably broader range of problems.

These individual difference patterns are present among both boys and girls, and among both middle-class suburban children and low-income inner-city children. The relative frequencies of children in each group also are comparable for boys and girls and for low-income and middle-income children (Kerkman & Siegler, 1993; 1997). Both of these findings are different than stereotypes would suggest. At least at this point in learning, individual difference patterns in mathematics are comparable among boys and girls and among low and middle income children.

These classifications of children's individual difference patterns are related to traditional ones. Roughly half of the not-so-good students in the Siegler (1988) study went on to be classified as having mathematical disabilities by fourth grade, versus none of the good students or perfectionists. However, the cognitive assessments go beyond the traditional ones in showing stylistic differences in math performance as well as knowledge-based ones.

As has often been noted (e.g., Geary, 1994), mathematical disabilities, as defined by poor performance in class and poor standardized test scores, constitutes a very serious problem in the U.S. Approximately 6% of children are labeled as having such disabilities. Like the not-so-good students in the previous description, these children have difficulty both in executing backup strategies and in retrieving correct answers. As first graders, they frequently use immature counting procedures (counting from one rather than from the larger addend), execute backup strategies slowly and inaccurately, and use retrieval rarely and inaccurately. By second grade, they use somewhat more sophisticated counting procedures, such as counting from the larger addend, and their speed and accuracy improve. However, they continue to have difficulty retrieving correct answers then and for years after (Geary, 1990; Geary & Brown, 1991; Goldman, Pellegrino, & Mertz, 1988; Jordan, Levine, & Huttenlocher, 1997). As they progress through school, these children encounter further problems in the many skills that build on basic arithmetic, such as multidigit arithmetic and algebra (Zawaiza & Gerber, 1993; Zentall & Ferkis, 1993).

Why do some children encounter such large problems with arithmetic? One reason is limited exposure to numbers before entering school. Many children labeled "mathematically disabled" come from poor families with little formal education. By the time children from such backgrounds enter school, they already are far behind other children in counting skill, knowledge of numerical magnitudes, and knowledge of arithmetic facts. Another key difference involves working memory capacity. Learning of arithmetic requires sufficient working memory capacity to hold the original problem in memory while computing the answer so that the problem and answer can be associated. However, children labeled as mathematically disabled cannot hold as much numerical information in memory as age peers (Geary, Bow-Thomas, & Yao, 1992; Koontz & Berch, 1996.) Limited conceptual understanding of arithmetic operations and counting adds further obstacles to these children's learning of arithmetic (Hitch & McAuley, 1991; Geary, 1994). Thus, mathematical disabilities reflect a combination of limited background knowledge, limited processing capacity, and limited conceptual understanding. All of these difficulties need to be addressed in order for such children to learn mathematics to a reasonably high level of proficiency.

Discovery and Insight

The previously-described findings with preschoolers' understanding of counting principles showed that even before children enter school, they think abstractly about certain mathematical concepts. However, it takes surprisingly long for this understanding to be expanded to other concepts, even ones that also pertain directly to understanding numbers. One such understanding that children discover in the first few years of elementary school is the *inversion principle*--the idea that adding and subtracting the same number leaves the original quantity unchanged. Understanding of this principle can be assessed through examining performance on problems of the form "a + b - b = ?" (e.g., 25 + 8 - 8 = ?). Children who solve such problems through applying the inversion principle would answer in the same amount of time regardless of the size of *b*, because they would not need to add and subtract it. In contrast, children who solve the problem by adding and subtracting *b* would take longer when *b* was large than when it was small, because adding and subtracting large numbers takes longer than adding and subtracting small ones.

Between 6 and 9 years, performance on a+b-b problems becomes faster. However, 9-year-olds, like 6-year-olds, take longer on problems where *b* is large than on ones where it is small (Bisanz & LeFevre, 1990; Stern, 1992). The improved speed on all problems appears due to improved procedural competence in addition and subtraction. The continuing difference between times on problems where *b* is large and ones where it is small suggests that neither 6- nor 9-year-olds have sufficient conceptual competence (i. e., understanding of the inversion principle) to consistently answer such problems without adding and subtracting. Not until age 11 do most children, like almost all adults, ignore the particular value of *b* and solve all problems equally quickly, thus demonstrating understanding of the inversion principle.

A related concept that also takes surprisingly long for children to understand is that of *mathematical equality*. Even third and fourth graders frequently do not understand that the equal sign means that the values on each side of it must be equal. Instead, they believe that the equal sign is simply a signal to execute an arithmetic operation. On typical problems such as 3+4+5=____, this misinterpretation does not cause any difficulty. However, on atypical problems, such as 3+4+5=____+5, the misunderstanding leads most third and fourth graders either to just add the numbers to the left of the equal sign, and answer "12," or to add all numbers on both sides of it, and answer "17" (Alibali & Goldin-Meadow, 1993; Goldin-Meadow, Alibali, & Church, 1993; Perry, Church, & Goldin-Meadow, 1988; 1992).

Research on how children solve mathematical equality problem indicates that they frequently make hand gestures that indicate knowledge that is not evident in their verbal statements. For example, some children who answer "12" and explain that they just added 3+4+5 also motion with their hands toward the 12 in ways that indicate equality between the two. Children who on a pretest show such discrepancies between their speech and their gestures subsequently learn more from instruction in how to solve these problems than do children whose gestures and speech on the pretest reflect the same understanding (Alibali & Goldin-Meadow, 1993; Goldin-Meadow, et al., 1993; Perry et al., 1988; 1992). When asked to evaluate videotapes of children solving such problems, teachers and other children rate more highly solutions that include advanced gestures than ones that do not, even when what the children say and write is identical (Garber, Alibali, & Goldin-Meadow, 1998). An implication is that to the extent possible under classroom conditions, teachers should attend to children's nonverbal gestures as well as their verbal statements as indicators of their understanding and readiness to learn.

One lesson that has emerged from many recent studies is that children discover new strategies and concepts when existing approaches are succeeding as well as when they are failing (e.g., Karmiloff-Smith, 1992; Kuhn et al., 1995;Miller & Aloise-Young, 1996; Siegler & Jenkins, 1989). We often assume that necessity is the mother of invention, and sometimes it is. However, children frequently generate new approaches on problems that they have previously solved using existing methods, and when thy have been succeeding on the preceding problems. One implication of this pervasive finding is that many discoveries do not require special "discovery learning" situations to be created. In the same way that adults often generate new ideas while in the shower, while driving, or while working on unrelated or minimally related problems, so do children.

Relations Between Conceptual and Procedural Knowledge

Throughout this century, instructional reform has oscillated between emphasizing mastering of facts and procedures on the one hand and emphasizing understanding of concepts on the other (Hiebert & Lefevre, 1986). Few today would argue that either type of mathematical knowledge should be taught to the exclusion of the other. Much less agreement exists, however, concerning the balance between the two that should be pursued or concerning how to design instruction that will inculcate both types of knowledge.

Multidigit addition and subtraction has proved to be an especially fruitful domain for studying the relations between conceptual and procedural knowledge. Children spend several years learning multidigit arithmetic. They must learn the carrying procedure for addition and the borrowing procedure for subtraction. Understanding these procedures requires understanding of the concept of place, that each position in a multidigit number represents a successively higher power of ten. It also requires understanding that a multidigit number can be represented in different ways, for example, 23 can be represented as 1 "10" and 13 "1's".

Many children have difficulty understanding place value, and, as noted earlier, they frequently use buggy procedures that reflect this lack of understanding. For example, second-graders often do not correctly carry when adding multidigit numbers (Fuson & Briars, 1990). Instead, they either write the two-digit sums beneath each column of single-digit addends (e. g., 568+778=121316) or ignore the carried values (e. g., 568+778=1236).

Although there are exceptions, procedural skill and conceptual understanding usually are highly correlated. One source of evidence for this view is cross-national studies. For example, comparisons of Korean and American elementary school children have revealed parallel national differences in conceptual and procedural knowledge of multidigit addition and subtraction. Fuson and Kwon (1992) asked Korean second and third graders to solve two and three digit addition and subtraction problems that require carrying or regrouping. Then the children were presented several measures of conceptual understanding: ability to identify correctly and incorrectly worked out addition and subtraction problems, to explain the basis of the correct procedure, and to indicate the place value of digits within a number. Almost all of the Korean children used correct procedures to solve the problems and also succeeded on all of the measures of conceptual understanding. Stevenson and Stigler (1992) reported similar procedural and conceptual competence in first through fifth graders in Japan and China.

On the other hand, a number of studies reviewed in Fuson (1990) indicated that American children, ranging from second to fifth grade, frequently lack both conceptual and procedural knowledge of multidigit addition and subtraction. Lack of conceptual understanding was evident in findings that almost half of third graders incorrectly identified the place value of digits within multidigit numbers (Kouba, Carpenter & Swafford, 1989; Labinowicz, 1985), and in findings that most second through fifth graders could not demonstrate or explain ten-for-one trading with concrete representations (Ross, 1986). Lack of procedural knowledge was evident in findings that children of these ages frequently erred while using paper and pencil to solve multidigit addition and subtraction problems (Brown & Burton, 1978; Fuson & Briars, 1990; Kouba et al., 1989; Labinowicz, 1985; Stevenson and Stigler, 1992). Taken together, these results suggest that conceptual and procedural knowledge are related; Asian children have both, and American children lack both.

Within the U. S., conceptual and procedural competence are also highly correlated. Second and third graders who correctly execute the subtraction borrowing procedure also are more accurate in detecting conceptual flaws in a puppet's subtraction procedures than are children who do not execute the subtraction algorithm consistently corrrectly (Cauley, 1988). Conceptual understanding of multidigit addition and subtraction and the ability to invent effective computational procedures are also positively correlated in first through fourth graders (Hiebert & Wearne, 1996).

This correlation leaves open the possibility that conceptual understanding could be causally related to children inventing adequate computational procedures, but also the possibility that knowing the correct procedure could be causally related to increased conceptual understanding (by allowing children to reflect on why the correct procedure is correct.) One relevant source of evidence is examination of the order in which individual children gain procedural and conceptual competence. It turns out that a substantial percentage of children first gain conceptual understanding and then procedural competence, but that another substantial percentage do the opposite (Hiebert & Wearne, 1996).

Studies aimed at improving teaching of multidigit addition and subtraction typically emphasize linking steps in the procedures to the concepts that support them. In general, these teaching techniques successfully increase both conceptual and procedural knowledge. Although not currently conclusive, they suggest that instruction that emphasizes conceptual understanding as well as procedural skill is more effective in building both kinds of competence than instruction that only focuses on procedural skill (Fuson & Briars, 1990; Hiebert & Wearne, 1996).

A question that remains, however, is which type of knowledge should be emphasized first. Many opinions have been offered on this topic, but until recently, no directly-relevant experimental evidence was available. A recent study by Rittle-Johnson and Alibali (in press), however, provides such evidence. They examined fifth graders' performance on mathematical equality problems of the form a+b+c=_____+c. Some randomly-selected children were presented conceptually-oriented instruction, other children were presented procedurally-oriented instruction, and yet others were presented neither. Then all children were presented practice solving problems, followed by a posttest that tested both conceptual and procedural knowledge. The conceptually-oriented instruction produced substantial gains in both kinds of knowledge; the procedurally-oriented instruction produced substantial gains in procedural knowledge and smaller gains in conceptual knowledge. To the degree that this result proves general, it suggests that conceptual instruction should be undertaken before instruction aimed at teaching procedures.

Cooperative Learning

Children discover new strategies not only while solving problems on their own, but also while working with others toward common goals. Some such problem solving involves scaffolding situations, in which a more knowledgeable person helps a less knowledgeable one to learn by providing a variety of kinds of help. Such scaffolding occurs in the context of parents helping their children, teachers helping their students, coaches helping their players, and more advanced learners helping less advanced ones (Freund, 1990; Gauvain, 1992; Rogoff, Ellis, & Gardner, 1984; Wood, Bruner, & Ross, 1976). The goal of such interactions is for the less knowledgeable learner to construct strategies that the more advanced already possess.

In other situations, equally-knowledgeable peers learn together. Such cooperative learning often enhances problem solving and reasoning relative to what children would achieve on their own (Gauvain & Rogoff, 1989; Kruger, 1992; Teasley, 1995; Webb, 1991). One particularly effective type of cooperative learning is reciprocal instruction, in which teachers read paragraphs with small groups of students and model such crucial metacognitive activities as summarizing, identifying ambiguities, asking questions, and predicting subsequent content. A recent review of 16 studies on reciprocal instruction (Rosenshine & Meister, 1994) indicated that results were generally positive with students ranging from fourth graders to adults, with both low-achieving and average students, with groups ranging from 2 to 23 students, and with either experimenters or classroom teachers as the instructors.

On the other hand, cooperative learning often does not result in increased learning, and at times leads to worse learning than trying to solve problems by oneself (Levin & Druyan, 1993; Russell, 1982; Russell, Mills, & Reiff-Musgrove, 1990; Tudge, 1992). As noted by Ann Brown (March 4, 1998, personal communication), perhaps the greatest expert on cooperative learning, designing effective cooperative learning situations requires at least as much engineering as does standard classroom instruction. Without such careful structuring, problems of freeloading and disorganization can lead to inferior learning. Thus, creating effective cooperative learning requires more than just assigning children to a group and telling them to work together on a problem or project.

Different types of collaborative organizations tend to have different effects not only on learning but also on instructional interactions. Damon and Phelps (1989) distinguished among three types of collaborative arrangements: peer tutoring, cooperative learning, and peer collaboration. Peer tutoring involves a child who is knowledgeable about a topic instructing another child who is less knowledgeable. Cooperative learning involves classrooms being divided into small groups or teams, usually 3-6 students of heterogeneous ability in each, trying to solve a problem or master a task. In a common variant of cooperative learning, the jigsaw method, each child becomes the group's expert on a particular part of the task, and task solutions require the contributions of all of the experts. Finally, peer collaboration involves a pair of novices working together to solve problems that neither could solve on their own initially. These arrangements tend to differ in the degree to which they promote equality among participants (higher in peer collaboration and cooperative learning than in peer tutoring) and in the degree to which discussions tend to be extensive and engaging (highest in peer collaboration). Damon and Phelps argued that the collaborative arrangements that generated the most productive instructional dialogs were those that encouraged joint problem solving and that discouraged competition among students.

How can the effectiveness of collaborations be improved? One way is to examine factors that differentiate successful from unsuccessful interactions. To obtain such information, Ellis, Siegler, and Klahr (1993) examined fifth graders solving decimal fraction problems of the form: "Which is bigger, .239 or .47"? As noted earlier, this seemingly simple task often causes children of this age considerable difficulty. In particular, they often misapply mathematical rules acquired while learning about whole numbers or common fractions, either consistently choosing the number with more digits as larger (the *Whole Number Rule*) or consistently choosing the number with fewer digits as larger (the *Fraction Rule*).

Ellis et al. found that children learned more if they worked with a partner during training than if they worked alone; that this benefit occurred only if the children were also provided feedback by the experimenter concerning which answer was right; that external feedback was just as critical for partners who started with different rules as it was for those who started with the same rule; and that social aspects of the interaction, as well as external feedback, influenced learning. One particularly important factor was the enthusiasm of the partner's reactions to the child's statements. In this context, enthusiasm meant strong interest in, rather than agreement with, the partner's ideas. Children whose partner reacted enthusiastically during the training session answered correctly more often on the posttest than did those whose partners showed less enthusiasm. Among children who worked with a partner and received feedback from the experimenter, the enthusiasm of the partner's reactions was the best single predictor of learning. The example illustrates that attention to both cognitive and social variables is crucial for successful cooperative learning.

Promoting Analytic Thinking and Transfer

Analytic thinking refers to a set of processes for identifying the causes of events. It is among the central goals of mathematics education. One reason is that analytic thinking is an inherently constructive process. Analysis demands that children actively think about the causes of events. It is possible to obtain a general sense of the typical course of events, or the way in which things work, without actively analyzing them. However, distinguishing features that usually accompany events from those that cause them to occur requires more active thinking. Thus, distinguishing between features that typically accompany the use of a particular mathematical problem solving technique, and features that are essential for the technique to apply, usually requires analysis of why the technique is appropriate or inappropriate.

Analytic reasoning is both cause and consequence of another useful quality: *purposeful engagement*. When children have a specific reason for wanting to learn about a topic, they are more likely to analyze the material so that they truly understand it. In this sense, analytic reasoning is a consequence of purposeful engagement. However, analytic reasoning also promotes purposeful engagement. Children who from the beginning of learning about a topic try to deeply understand it become more engaged in learning it than children who accept what they are told without thinking about it.

A third way in which analytic thinking is central is in promoting *transfer*. When children are actively engaged in understanding why things work the way they do, transfer follows naturally and without great effort. In contrast, when understanding stays close to the surface, and does not penetrate underneath, transfer is unlikely (Brown, 1997). The problem is that such passive learners lack ways of distinguishing the core information from the incidental details. Thus, encouraging learners to more often reason analytically will also create learners who transfer what they learn to new situations.

A variety of types of evidence attest to the importance of such explanatory activities. For both adults and children, students who ask themselves more questions about the meaning of a textbook as they are reading it learn more from their reading than do children who read without asking many questions. This has been shown for learning of both physics and computer programming (Chi et al., 1989; Pirolli & Recker, 1994). The quality as well as the quantity of explanations that children generate while reading is related to their learning. For example, when the best learners study example problems, they are especially likely to connect particular aspects of the examples to particular statements in the text (Pirolli & Bielaczyc, 1989).

How can such analytic thinking be encouraged? One effective way is to ask children to explain the correct conclusions or answers of other people. Children as young as 5 years benefit when they are asked to answer a difficult problem, answer it, are told the correct answer, and then are asked, "How do you think I knew that" (Siegler, 1995). This procedure combines advantages of discovery learning with those of didactic methods. Like discovery-learning approaches, it promotes active engagement with the task, since children have to generate the underlying logic for themselves. Like didactic methods, it is efficient; rather than going down blind alleys, children spend their time thinking about the logic that led to desired conclusions. An added advantage of this approach is that it can be applied to a very wide variety of problems. It is possible to ask about almost any conclusion, "How do you think I knew that" (or "Why do you think I think that".) Encouraging children to explain other people's reasoning in many contexts may lead children to internalize such an analytic stance to the point where they ask such questions reflexively, even when not prompted to do so.

Asking children to explain both why correct answers are correct and why incorrect answers are incorrect may be even more effective than just asking them to explain why correct answers are correct. Such activities are featured in Japanese classrooms, and are associated with excellent levels of math achievement in that country (Stigler & Perry, 1990). They also are effective with U. S. children. In a recent experiment on understanding of mathematical equality, children were randomly assigned to one of three conditions: explain both why correct answers are correct and why incorrect answers are wrong, just explain why correct answers are correct, or just try to solve mathematical equality problems and receive feedback (Siegler, in preparation). Asking children to explain both why correct answers are right and why incorrect answers are wrong led to greater learning than just asking the former type of question. Especially encouraging, it greatly increased transfer to problems that were superficially dissimilar to the originally-presented ones.

Teaching children computer programming skills has been proposed as another means of promoting transfer. In particular, advocates of providing such experience have contended that it would produce not only skill at programming, but also enhanced general problem-solving ability and analytic skills. In one notable effort in this direction, Papert (1980) designed the LOGO language with the goal of helping children acquire such broadly useful skills as dividing problems into their main components, identifying logical flaws in one's thinking, and generating well-thought-out plans.

When learned in standard ways, LOGO has proved insufficient to meet these goals. However, *mediated instruction*, in which LOGO is taught with an eye toward building transferable skills, has been quite successful in producing the desired effects (Klahr & Carver, 1988; Lehrer & Littlefield, 1991; 1993; Littlefield, Delclos, Bransford, Clayton, & Franks, 1989). Like conventional instruction in computer programming, mediated instruction involves teachers demonstrating to students how to use commands and concepts and providing them with feedback on their attempts to use them. However, mediated instruction also involves teachers explicitly noting when particular commands and programs illustrate general programming concepts and drawing explicit analogies between the reasoning used to program and to solve problems in other contexts.

Such mediated instruction has produced a variety of kinds of desirable transfer. For example, Klahr and Carver (1988) demonstrated that mediated instruction in LOGO can create debugging skills that are useful outside as well as inside the LOGO context. Their instructional program was based on a detailed task analysis of debugging. Within this analysis, the debugging process begins with the debugger determining the outcome that a procedure yields and observing if and how its results deviate from what was planned (for example by running a computer program and examining its output). Following this, the debugger describes the discrepancy between desired and actual outcomes, and hypothesizes types of bugs that might be responsible. The next step is to identify parts of the program that could conceivably produce the observed bug. This step demands dividing the program into components, so that specific parts of the program are identified with specific functions. Following this, the debugger first checks the relevant parts of the program to see which, if any, fail to produce the intended results; then rewrites the faulty component; and then runs the debugged program to determine if it now produces the desired output. The 8- to 11-year-olds who received this instruction took barely half as long to solve LOGO debugging problems as children who were not presented it. They also improved their general problem solving skills in areas outside of programming, in particular revision of essays. The improvement seemed due to the children applying the skills taught in the program: analyzing the nature of the original discrepancy from the anticipated results, hypothesizing possible causes, and focusing their search on relevant parts of the instructions, rather than simply checking them line-by-line.

Conclusions

This paper summarizes a number of empirical findings and theoretical conclusions about children's mathematics learning. Translating these findings and conclusions into improved instructional practices, however, will take a considerable amount of work. Stigler and Hiebert's (1998) description of the Japanese emphasis on *continuous improvement* in teaching points toward a process that seems essential in U.S. classrooms as well. The process that they describe involves groups of teachers working together to perfect the way in which they teach particular concepts and procedures. Neither controlled scientific experimentation nor theoretical analyses automatically translate into prescriptions for classroom instruction. They can provide useful frameworks for thinking about teaching and learning, can indicate sources of difficulty that children encounter in learning particular skills and concepts, and can demonstrate potentially effective instructional procedures. However, a process of translation into the particulars of each classroom context is necessary for even the most insightful frameworks and the most relevant findings to be utilized in ways that improve learning. Both institutional support for such continuous improvement and teacher dedication to meeting this goal are essential if research is to lead to superior instruction.

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References

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. Cognitive Psychology, 25, 468-573.

Anderson, J. R., Reder, L., M., & Simon, H. A. (1997). Radical constructivism and cognitive psychology. In D. Ravitch (Ed.) Brookings papers on education policy. Washington, DC: Brookings Institution Press.

Antell, S. E., & Keating, D. P. (1983). Perception of numerical invariance in neonates. Child Development, 54, 695-701.

Ashcraft, M. H. (1987). Children's knowledge of simple arithmetic: A developmental model and simulation. In J. Bisanz, C. J. Brainerd, & R. Kail (Eds.), Formal methods in developmental psychology. New York: Springer-Verlag.

Bisanz, J., & LeFevre, J. (1990). Mathematical cognition: Strategic processing as interactions among sources of knowledge. In D. P. Bjorklund (Ed.), Children's strategies: Contemporary views of cognitive development. Hillsdale, NJ: Erlbaum.

Brown, J.S., & R.B. Burton (1978). Diagnostic models for procedural bugs in basic mathematical skills. Cognitive Science, 2, 155-192.

Brown, A. L. (1997). Transforming schools into communities of thinking and learning about serious matters. American Psychologist, 52, 399-413.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1981). Results from the second mathematics assessment of the National Assessment of Educational Progress. Washington, DC: National Council of Teachers of Mathematics.

Cauley, K. M. (1988). Construction of logical knowledge: Study of borrowing in subtraction. Journal of Educational Psychology, 80, 202-205.

Chi, M. T. H., & Klahr, D. (1975). Span and rate of apprehension in children and adults. Journal of Experimental Child Psychology, 19, 434-439.

Chi, M. T. H., Bassok, M., Lewis, M., Reimann, P., & Glasser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. Cognitive Science, 13, 145-182.

Chi, M. T. H., de Leeuw, N., Chiu, M.-H., & LaVancher, C. (1994). Eliciting self-explanations improves understanding. Cognitive Science, 18, 439-477.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. Journal for Research in Mathematics Education, 13, 16-30.

Damon, W., & Phelps, E. (1989). Critical distinctions among three approaches to peer education. International Journal of Educational Research, 13, 9-20.

Ellis, S., Klahr, D., & Siegler, R. S. (1993, March). Effects of feedback and collaboration on changes in children's use of mathematical rules. Paper presented at the meeting of the Society for Research in Child Development, New Orleans.

Freund, L. S. (1990). Maternal regulation of children's problem solving behavior and its impact on children's performance. Child Development, 61, 113-126.

Fuson, K.C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. Cognition and Instruction, 7, 343-403.

Fuson, K. C., & D. Briars (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. Journal for Research in Mathematics Education, 21, 180-206.

Fuson, K. C., & Kwon, Y. (1992). Korean children's understanding of multidigit addition and subtraction. Child Development, 63, 491-506.

Garber, P., Alibali, M. W., & Goldin-Meadow, S. (1998). Knowledge conveyed in gesture is not tied to the hands. Child Development, 69, 75-84.

Gauvain, M. (1992). Social influences on the development of planning in advance and during action. International Journal of Behavioral Development, 15, 377-398.

Gauvain, M., & Rogoff, B. (1989). Collaborative problem solving and children's planning skills. Developmental Psychology, 25, 139-151.

Geary, D. C. (1990). A componential analysis of an early learning deficit in mathematics. Journal of Experimental Child Psychology, 49, 363-383.

Geary, D. C. (1994). Children's mathematical development: Research and practical implications. Washington, DC: American Psychological Association.

Geary, D. C., Bow-Thomas, C. C., & Yao, Y. (1992). Counting knowledge and skill in cognitive addition: A comparison of normal and mathematically disabled children. Journal of Experimental Child Psychology, 54, 372-391.

Geary, D. C., & Brown, S. C. (1991). Cognitive addition: Strategy choice and speed-of-processing differences in gifted, normal, and mathematically disabled children. Developmental Psychology, 27, 398-406.

Geary, D. C., Fan, L., Bow-Thomas, C., Siegler, R. S. (1993). Even before formal instruction, Chinese children outperform American children in mental addition. Cognitive Development 8, 517-529.

Gelman, R., & Gallistel, C. R. (1978). The child's understanding of number. Cambridge, MA: Harvard University Press.

Goldin-Meadow, S., Alibali, M. W., & Church, R. B. (1993). Transitions in concept acquisition: Using the hand to read the mind. Psychological Review, 100, 279-297

Goldman, S. R., Pellegrino, J. W., & Mertz, D. L. (1988). Extended practice of basic addition facts: Strategy changes in learning disabled students. Cognition and Instruction, 5, 223-265.

Griffin, S., Case, R., & Siegler, R. S. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), Classroom lessons: Integrating cognitive theory and classroom practice (pp. 25-49). Cambridge, MA: MIT Press/Bradford Books.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In Conceptual and procedural knowledge: The case of mathematics, J. Hiebert (ed.), 29-57. Hillsdale, NJ: Erlbaum.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding and skill in multidigit addition and subtraction. Cognition and Instruction, 14, 251-283.

Hitch, G. J., & McAuley, E. (1991). Working memory in children with specific arithmetical learning disabilities. British Journal of Psychology, 82, 375-386.

Jordan, N. C., Levine, S. C., & Huttenlocher, J. (1995). Calculation abilities in young children with different patterns of cognitive functioning. Journal of Learning Disabilities, 28, 53-64.

Karmiloff-Smith, A. (1992). Beyond modularity: A developmental perspective on cognitive science. Cambridge, MA: MIT.

Kerkman, D. D., & Siegler, R. S. (1993). Individual differences and adaptive flexibility in lower-income children's strategy choices. Learning and Individual Differences, 5, 113-136.

Kerkman, D. D., & Siegler, R. S. (1997). Measuring individual differences in children's addition strategy choices. Learning and Individual Differences, 9, 1-18.

Klahr, D., & Carver, S. M. (1988). Cognitive objectives in a LOGO debugging curriculum: Instruction, learning, and transfer. Cognitive Psychology, 20, 362-404.

Koontz, K. L., & Berch, D. B. (1996). Idntifying simple numerical stimuli: Processing inefficiencies exhibited by arithmetic learning disabled children. Mathematical Cognition, 2, 1-23.

Kouba, V. L., Carpenter, T. P., & Swafford, J. O. (1989). Number and operations. In M. M. Lindquist (Ed.), Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress. Reston, VA: National Council of Teachers of Mathematics, Inc.

Kuhn, D., Garcia-Mila, M., Zohar, A., & Andersen, C. (1995). Strategies of knowledge acquisition. Monographs of the Society for Research in Child Development, 60, Serial No. 245.

Kruger, A. C. (1992). The effect of peer and adult-child transactive discussions on moral reasoning. Merrill-Palmer Quarterly, 38, 191-211.

Kuhn, D., Garcia-Mila, M., Zohar, A., & Andersen, C. (1995). Strategies of knowledge acquisition. Monographs of the Society for Research in Child Development, 60, Serial No. 245.

Labinowicz, E. (1985). Learning from children: New beginnings for teaching numerical thinking. Menlo Park, CA: Addison-Wesley.

Lehrer, R., & Littlefield, J. (1991). Misconceptions and errors in LOGO: The role of instruction. Journal of Educational Psychology, 83, 124-133.

Lehrer, R., & Littlefield, J. (1993). Relationships among cognitive components in LOGO learning and transfer. Journal of Educational Psychology, 85, 317-330.

Levin, I., & Druyan, S. (1993). When sociocognitive transaction among peers fails: The case of misconceptions in science. Child Development, 63, 1571-1591.

Littlefield, J., Delclos, V. R., Bransford, J. D., Clayton, K. N., & Franks, J. J. (1989). Some prerequisites for teaching thinking: Methodological issues in the study of LOGO programming. Cognition and Instruction, 6, 331-366.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J. S. Brown (Eds.), Intelligent tutoring systems. New York: Academic Press.

Miller, P., & Aloise-Young, P. (1996). Preschoolers' strategic behaviors and performance on a same-different task. Journal of Experimental Child Psychology, 60, 284-303.

Papert, S. (1980). Mindstorms: Children, computers, and powerful ideas. New York: Basic Books.

Perry, M., Church, R. B., & Goldin-Meadow, S. (1988). Transitional knowledge in the acquisition of concepts. Cognitive Development, 3, 359-400.

Perry, M., Church, R. B., & Goldin-Meadow, S. (1992). Is gesture/speech mismatch a general index of transitional knowledge? Cognitive Development, 7, 109-122.

Pirolli, P., & Bielaczyc, K. (1989). Empirical analyses of self-explanation and transfer in learning to program. In Proceedings of the 11th Annual Conference of the Cognitive Science Society. Hillsdale, NJ: Erlbaum.

Pirolli, P., & Recker, M. (1994). Learning strategies and transfer in the domain of programming. Cognition and Instruction, 12, 235-275.

Resnick, L. B., Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding algebra. In J. A. Sloboda & D. Rogers (Eds.), Cognitive processes in mathematics. Oxford: Clarendon Press.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. Journal of Research in Mathematics Education.

Rittle-Johnson, B., & Alibali, M. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? Journal of Educational Psychology, 91, 175-189.

Rogoff, B., Ellis, S., & Gardner, W. P. (1984). Adjustment of adult-child instruction according to child's age and task. Child Development, 20, 193-199.

Rosenshine, B., & Meister, C. (1994). Reciprocal teaching: A review of the research. Review of Educational Research, 64, 479-530.

Ross, S. H. (1986, April). The development of children's place-value numeration concepts in grades two through five. Paper presented at the annual meeting of the

American Educational Research Association, San Francisco.

Russell, J. (1982). Cognitive conflict, transmission and justification: Conservation attainment through dyadic interaction. Journal of Genetic Psychology, 140, 283-297.

Russell, J., Mills, I., & Reiff-Musgrove, P. (1990). The role of symmetrical and asymmetrical social conflict in cognitive change. Journal of Experimental Child Psychology, 49, 58-78.

Schauble, L. (1996). The development of scientific reasoning in knowledge-rich contexts. Developmental Psychology, 32, 102-119.

Siegler, R. S. (1984). Research on learning. In T. Romberg and D. Stewart (Eds.), School mathematics: Options for the 1990s, 79-84.

Siegler, R. S. (1988). Individual differences in strategy choices: Good students, not-so-good students, and perfectionists. Child Development, 59, 833-851.

Siegler, R. S. (1995).How does change occur: A microgenetic study of number conservation. Cognitive Psychology, 28, 255-273.

Siegler, R. S. (2002). Microgenetic studies of self-explanations. In N. Granott & J. Parziale (Eds.), Microdevelopment: Transition processes in development and learning (pp. 31-58). New York: Cambridge University Press.

Siegler, R. S., & Jenkins, E. A. (1989). How children discover new strategies. Hillsdale, NJ: Erlbaum.

Siegler, R. S., & Robinson, M. (1982). The development of numerical understandings. In H. W. Reese & L. P. Lipsitt (Eds.), Advances in child development and behavior: Vol. 16, New York: Academic Press.

Silver, E. A. (1983). Probing young adults' thinking about rational numbers. Focus on Learning Problems in Mathematics, 5, 105-117.

Simon, T. J., Hespos, S. J., & Rochat, P. (1995). Do infants understand simple arithmetic: A replication of Wynn (1992). Cognitive Development, 10, 253-269.

Sleeman, D. H. (1985). Basic algebra revised: A study with 14-year-olds. International Journal of Man-Machine Studies, 22, 127-149.

Starkey, P., & Cooper, R. S. (1980). Perception of numbers by human infants. Science, 210, 1033-1035.

Starkey, P., Spelke, E. S., & Gelman, R. (1990). Numerical abstraction by human infants. Cognition, 36, 97-128.

Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. Contempory Educational Psychology, 17, 266-277.

Stevenson, H. W., & Stigler, J. W. (1992). The learning gap: Why our schools are failing and what we can learn from Japanese and Chinese education. New York: Summit Books.

Stigler, J. W., & Hiebert, J. (1998). Understanding and improving classroom mathematics instruction: An overview of the TIMSS Video Study. Unpublished manuscript.

Stigler, J. W., & Perry, M. (1990) Mathematics learning in Japanese, Chinese, and American classrooms. In J. W. Stigler, R. A. Shweder, & G. Herdt (Eds.), Cultural psychology: Essays on comparative human development. New York: Cambridge University Press.

Strauss, M. S., & Curtis, L. E. (1984). Development of numerical concepts in infancy. In C. Sophian (Ed.), The origins of cognitive skills. Hillsdale, NJ: Erlbaum.

Teasley, S. D. (1995). The role of talk in children's peer collaborations. Developmental Psychology, 31, 207-220.

Tudge, J. (1992). Processes and consequences of peer collaboration: A Vygotskian analysis. Child Development, 63, 1364-1379.

van Loosbroek, E., & Smitsman, A. W. (1990). Visual perception of numerosity in infancy. Developmental Psychology, 26, 916-922.

Webb, N. (1991). Task-related verbal interaction and mathematics learning in small groups. Journal for Research in Mathematics Education, 22, 366-389.

Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. Journal of Child Psychology and Psychiatry, 17, 89-100.

Wynn, K. (1992). Addition and subtraction by human infants. Nature, 358, 749-750.

Zawaiza, T. R., & Gerber, M. (1993). Effects of explicit instruction on math word-problem solving by community college students with learning disabilities. Learning Disability Quarterly, 16, 64-79.

Zentall, S. S., & Ferkis, M. A. (1993). Mathematical problem solving for youth with ADHD, with and without learning disabilities. Learning Disability Quarterly, 16, 6-18.