Strategy Choice Procedures and the Development of Multiplication Skill

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Many intelligent strategy choices may be accomplished through relatively low-level cognitive processes. This article describes a detailed model of how such "mindless" processes might lead to intelligent choices of strategies in one common situation: that in which people need to choose between stating a retrieved answer and using a backup strategy. Several experiments testing the model's applicability to children's single-digit multiplication are reported. These include tests of predictions about when different strategies are used and how early experience shapes later performance. Then, the sufficiency of the model to generate both performance at any one time and changes in performance over time is tested through the medium of a running computer simulation of children's multiplication. The simulation acquires a considerable amount of multiplication knowledge, and its learning and performance parallel those of children in a number of ways. Finally, several implications of the model for understanding cognitive self-regulation and cognitive development are discussed.

One of the deepest mysteries about cognition is how people decide what to do. It is clear that people often know many strategies for solving a problem; it is far less clear how they choose among them. The main purposes of this article are to describe a model of how people make one common strategy choice, to examine how the strategy choice process influences both immediate performance and subsequent learning, and to demonstrate the model's usefulness for explaining within a single framework a wide variety of findings on children's multiplication.

How strategies are chosen, and cognitive self-regulation in general, have received particular attention in theories of cognitive development. Such constructs as Piaget's (1976) reflective abstraction, Vygotsky's (1934) reflective consciousness, Binet's (1909) autocratism, Bruner's (1973) autonomous regulation, Flavell's (1979) metacognition, Sternberg's (1985) metacomponents, and Case's (1985) executive processes have all been motivated, at least in part, by the goal of explaining how children decide what to do.

Despite this widespread theoretical agreement concerning the importance of self-regulatory mechanisms, understanding of them has been slow in coming. The aforementioned constructs aptly describe the products of self-regulation but only hint at the cognitive processes through which self-regulation is accomplished.

How might self-regulatory mechanisms, in particular those involved with strategy choices, function? One possibility, associated with the construct of metacognition, is that strategy choices depend on explicit knowledge of strategies, capacities, and problem characteristics. For example, Kuhn (1984) contended, "In order to select a strategy as appropriate for solving a particular problem, the individual must understand the strategy, understand the problem, and understand how the problem and strategy intersect or map onto another" (p. 165).

As a number of researchers have noted, however, such "mindful" approaches now seem less promising as a general model of self-regulation than they once did. They have encountered both theoretical and empirical difficulties. Theoretically, they have been vague about the mechanisms by which they operate (Brown, Bransford, Ferrara, & Campione, 1983; Chi & Ceci, 1987; Sternberg & Powell, 1983; Wellman, 1983). Empirically, it has proved difficult to establish clear relations between cognition and explicit, stabile metacognitive knowledge (Brown & Reeve, 1986; Cavanaugh & Perlmutter, 1982; Sternberg & Powell, 1983; Wellman, 1983).

The present approach offers an alternative to approaches that assume explicit top-down regulation of strategy choices. Its basic assumption is that people make at least some of their strategy choices without reference to explicit knowledge of capacities, strategies, and problem characteristics. Rather than metacognition regulating cognition, cognitive representations and processes are assumed to be organized in such a way that they yield adaptive strategy choices without any direct governmental process.

The particular strategy choice focused on in this article is whether to state a retrieved answer or to use a backup strategy. A backup strategy is a sequence of cognitive operations, other than those involved in retrieval, that can generate correct answers to a class of problems. The choice between retrieval and backup strategies is an extremely common one. Following
are four examples from children's everyday lives: (a) When
adding numbers, children must choose between stating a
retrieved answer and using a backup strategy such as counting
fingers; (b) when spelling a word, they must choose between
writing a retrieved spelling and using a backup strategy such
as sounding out the word; (c) when looking up a phone
number, they must choose between just trying to remember
the number until dialing and using a backup strategy such as
rehearsal; and (d) when asked to name the sixth president,
they must choose between stating a retrieved answer and
using a backup strategy such as looking for the name in their
textbook.

The rest of this article examines how children could gen-
erate intelligent strategy choices in such situations without
rationally calculating the advantages and disadvantages of
each strategy. The first section describes a model of strategy
choice and how it might apply to children's single-digit mul-
tiplication. The next section tests the model's applicability to
children's multiplication through empirical experiments,
reanalyses of previous investigators' data, and a computer
simulation of learning and performance in this area. The final
section examines the implications of the model for a general
understanding of cognitive self-regulation and cognitive de-
velopment.

The Distribution of Associations Model

Siegler and Shrager (1984) developed the distribution of
associations model to account for strategy choices in pre-
schoolers' addition and subtraction. The basic organization
of the model, the strategy choice mechanism, and the factors
influencing acquisition are hypothesized to apply to many
tasks, among them older children's multiplication. The model
can be viewed in terms of a representation, a process, and a
basic learning mechanism.

Representation

Children are hypothesized to represent information about
specific problems (e.g., 6 × 4) in an associative network. The
network's main feature is associations between each problem
and possible answers, both correct and incorrect. Associations
across arithmetic operations (e.g., addition and multiplication)
also are hypothesized to be present in the network.

The representations of individual problems can be classified
along a dimension of the peakedness of their distributions of
associations. In a peaked distribution, such as that for 3 × 5
in Figure 1, most associative strength is concentrated in a
single answer, ordinarily the correct answer. At the other

![Graph showing associative strength for 3 × 5 and 6 × 9]

*Figure 1.* A peaked distribution of associations for 3 × 5 (dashed lines connecting squares) and a flat
distribution for 6 × 9 (solid lines connecting circles). Values for each answer reflect the percentage of
children who stated that answer in the retrieval-required experiment (Experiment 3). Thus, the
associative strength connecting the problem 3 × 5 and the answer 8 was .07 because 7% of the children
in Experiment 3 stated that answer on that problem.
extreme, in a flat distribution, such as that for $6 \times 9$ in Figure 1, associative strength is dispersed among several answers, with none of them forming a strong peak. As discussed in detail later, the peakedness of a problem's distribution is hypothesized to be a key determinant of its percentage of errors, length of solution times, and the strategies that are used on it.

Process

The process that operates on this representation involves three sequential phases, any one of which can produce an answer and thus terminate the process: retrieval, elaboration of the representation, and application of an algorithm. The existence of the three phases is constant across all distributions of associations models, though the particular elaborations and algorithms are task specific.

The hypothesized retrieval procedure is similar to those proposed in several recent models of memory (e.g., Anderson, 1983; Gillund & Shiffrin, 1984). As shown in Figure 2, at the outset of the retrieval phase, two parameters are set: a confidence criterion and a search length. The confidence criterion is a value that must be exceeded by the associative strength of a retrieved answer for the child to state that answer. The search length indicates the maximum number of retrieval efforts that the child will make. On each trial, the values of the confidence criterion and search length are set randomly, so that they are independent of each other and independent of their values on previous trials.

Once these two parameters are set, the child retrieves an answer. The probability of any given answer being retrieved on a particular retrieval effort is proportional to the associative strength of that answer relative to the associative strengths of all answers to the problem. Thus, in the Figure 1 example, the probability of retrieving 15 as the answer to $3 \times 5$ would be .80, because the associative strength connecting $3 \times 5$ and 15 is .80 and because the associative strengths for all answers to $3 \times 5$ in this example sum to 1.00.

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*Figure 2. Process operating on distribution of associations.*
If the associative strength of the retrieved answer exceeds the confidence criterion, the child states that answer. Otherwise, if the number of searches that has been conducted on the trial is within the permissible search length, the child again retrieves an answer, compares it to its associative strength with the confidence criterion, and advances it as the solution if its associative strength exceeds the criterion. Retrieval efforts continue as long as the associative strength of each retrieved answer is below the confidence criterion and the number of searches does not exceed the search length.

On a percentage of trials in which the maximum number of searches has been made and no answer stated, an alternate form of retrieval, the sophisticated guessing approach, is used. This involves dropping the confidence criterion to 0, retrieving one last answer, and stating whichever answer is retrieved. As in the standard form of retrieval, the probability of retrieving any given answer is proportional to the answer's associative strength. This implies that as long as the correct answer's associative strength is greater than that of any other answer, it will have a greater probability of being stated than any other answer (thus the description sophisticated guessing approach). Intuitively, the process can be likened to trying to spell a word such as asymmetry, not having great confidence that any retrieved spelling is correct, and writing out one of them anyway.

If children do not state an answer obtained through the standard retrieval or sophisticated guessing route, they may elaborate their representation. In multiplication, when pencil and paper are available, these elaborations usually involve writing the problem. The written elaboration may itself be associated with possible answers to the problem. For example, seeing \(3 \times 5\) on a page may trigger associations with 15 above and beyond those triggered by hearing the problem. (For evidence that visual associations influence arithmetic performance, see Gonzalez & Kulers, 1982.) Having written the problem, the child again retrieves an answer. As previously, if the confidence criterion is exceeded by the answer's associative strength, which now includes the visual association as well as the original problem-answer association, the child states it as the product.

If no answer has been stated, the child proceeds to the third phase. This is an algorithmic process, so labeled because in it, the elaborated representation is operated on in a way that always produces the correct answer if the algorithm is executed correctly. In multiplication, one algorithm that children can use is repeated addition, in which they write one multiplicand the number of times indicated by the other, sum the multiplicands, and state the sum as the answer.

**Acquisition**

How might children acquire the hypothesized process and representation? First, consider the process. Retrieval seems to be innate to human beings and to inherently involve qualities similar to confidence criteria and search lengths. Mervis and Canada (1982) found that even 1-year-olds often will not state an answer that is suggested to them if they do not believe the answer is correct. Such resistance suggests that they possess a standard, akin to a confidence criterion, for stating an answer. Very young children also will visibly give up after trying for a time to retrieve an answer, implying a criterion, like a search length, for stopping retrieval efforts.

Whereas retrieval seems to be innate, elaborations of representations and solution algorithms seem to be acquired through direct instruction. For example, children's most frequent multiplication algorithm, repeated addition, is taught early and at considerable length both by textbooks (e.g., Eicholz, O'Daffer, & Fleener, 1985) and by teachers.

Now consider acquisition of the knowledge in the representation. The basic assumption here is that people associate whatever answer they state, correct or incorrect, with the problem on which they state it. This assumption reduces the issue of what factors lead children to develop a particular distribution of associations on each problem to the issue of what factors lead them to state particular answers on each problem.

The same three factors hypothesized to influence formation of the associative network in preschoolers' addition and subtraction are also hypothesized to influence its formation in older children's multiplication. One factor is the answers generated by backup strategies. In both addition and subtraction, the more operations needed to use a backup strategy on a problem, the more likely children are to err when using it (Siegel, 1986; Siegel & Shragar, 1984). This was expected to be the case in multiplication as well. For example, solving the problem 6 × 9 through repeated addition should produce more errors than solving 5 × 3 in the same way. Further, backup strategies are hypothesized to produce characteristic errors that become associated with problems. For example, repeated addition should produce two types of errors: errors in which a multiplicand is added too many or too few times (e.g., 8 × 4 = 36) and errors in which a small addition error is made (e.g., 8 × 4 = 33).

The second factor believed to affect acquisition of the associative network is the influence of related numerical operations. These influences are particularly evident in children's errors. For example, Siegel and Shragar (1984) found that preschoolers' most frequent retrieval error on addition problems in which the second addend is greater than the first is consistently the number one greater than the second addend (3 + 4 = 5 and 3 + 5 = 6). Such errors may arise through preschoolers confusing addition with the related and (to them) better known operation of counting. Under time pressure, adults make similar errors in confusing multiplication with addition (e.g., 4 × 3 = 7; Winkelman & Schmidt, 1974).

The third factor hypothesized to influence acquisition of multiplication is frequency of problem presentation. Ashcraft (1987) and Siegel and Shragar (1984) found that frequency of problem presentation by parents and textbooks influences the relative difficulty of addition problems; the same may be the case in multiplication.

**Relations to Alternative Approaches to Arithmetic**

In recent years, researchers have proposed a number of models of children's and adults' single-digit multiplication (Ashcraft, 1987; Campbell, 1987; Campbell & Graham, 1985; Geary & Widaman, 1987; Miller, Perlmuter, & Keating, 1984; Stazyk, Ashcraft, & Hamann, 1982). These models share several features, particularly the assumption that knowl-
edge of multiplication can be represented as an associative network linking problems and answers. In the present context, the models of children's multiplication are the most directly relevant and therefore are examined in greatest detail.

Ashcraft (1987) proposed that children's knowledge of single-digit multiplication is stored in a memory representation akin to a fact table in an arithmetic book. Within the table, the digits 0 through 9 are the column and row heads; the answer to each problem is located at the intersection of the two multiplicands. Retrieval is accomplished by a spread of activation from each of the multiplicands to their intersection. Retrieving the answer to larger number problems is said to take longer than retrieving answers to smaller number problems because it takes longer in such cases for activation to spread down the columns and across the rows. Retrieval is always used on problems in which a sufficiently strong association exists between the problem and its correct answer. A backup strategy is always used on problems in which the problem-answer association is not sufficiently strong. Learning is produced by two aspects of practice: the relative frequency with which children encounter problems in textbooks and the cumulative frequency of practice of all problems over grade levels.

Campbell and Graham's (1985) model resembles Ashcraft's (1982) in many ways. Knowledge of multiplication is represented as a network of problem-answer associations. Retrieval is accomplished through the spread of activation. Learning occurs to a large degree through amount of practice. Two differences between the two models are that Campbell and Graham did not posit any use of backup procedures by children, and that they suggested that the earlier a problem begins to be practiced, the higher the degree of eventual mastery.

The distribution of associations model shares several features with these approaches: the representation of information within an associative network, the existence of associations across arithmetic operations, and the role of frequency of problem presentation as an influence on learning. However, important differences exist as well. One difference is in the hypothesized relation between early and later performance. Within the distribution of associations model, but not within any other model of children's or adults' arithmetic, earlier answers to problems (both correct and incorrect) are assumed to shape the later associative network. This child-is-father-to-the-man hypothesis led to the prediction that 8- and 9-year-olds' multiplication answers would parallel in considerable detail the associative networks of adults, a prediction that was tested in the present study.

A second, related difference is in the connection between backup strategies and retrieval. Within the distribution of associations model, backup strategies are critical shapers of retrieval performance through their influence on the associative network. Examination of the performance generated by use of backup strategies allows a single explanation for characteristics of retrieval that either are not explained or are viewed as independent in other models. For example, close associations have been found between problems and two types of incorrect answers: multiples of a multiplicand, such as $6 \times 8 = 56$, and close misses, such as $6 \times 8 = 46$. Ashcraft (1987) noted that his model accounts for the associations between problems and incorrect multiples, such as $6 \times 8 = 56$, but not for associations with close misses, such as $6 \times 8 = 46$. As noted already, within the distribution of associations model, both phenomena reflect the types of errors generated by backup strategies becoming associated with problems and therefore interfering with retrieval of the correct answer.

A third difference concerns the models' approaches to variability in answers and strategies. As McCluskey, Sokol, Cohen, and Ijiri (1986) noted, previous models of multiplication do not indicate how atypical answers arise. They also do not indicate how people come to use one strategy one day yet a different strategy the next (Siegel, 1986). In contrast, such variability of answers and strategies is an integral part of the distribution of associations model. With regard to the variability of answers, all answers associated with a problem have some probability of being retrieved; even weakly associated ones will at times exceed the confidence criterion on the trial and be stated. With regard to the variability of strategies, sometimes an answer is retrieved that has sufficient associative strength to exceed the confidence criterion on that trial, leading to use of retrieval; sometimes no such answer is retrieved, leading to use of a backup strategy. In sum, it appeared that the distribution of associations model had the potential to account for a broader range of phenomena than previous models of multiplication.

Before describing the several empirical and modeling investigations in this study, it may be helpful for me to provide an overview. Experiment 1 was designed to establish a large body of multiplication phenomena for the model to explain and to test two predictions of the model concerning when backup strategies are used. The next three parts of the study, two experiments and an analysis of problems presented in arithmetic workbooks, examined whether three factors hypothesized by the model to be influential—backup strategies, frequency of problem presentation, and the retrieval process—could explain the phenomena attributed to them. Following this, reanalyses of previous investigators' data were used to test a prediction about long-term influences of early learning. Finally, a computer simulation was implemented to demonstrate the sufficiency of the model to account for the phenomena observed in the present and previous experiments.

Experiment 1: Standard Multiplication

There were several reasons for trying to extend the distribution of associations model to the task of children's multiplication. Probably the most important was demonstrating the model's applicability to older children's strategy choices. Previous applications to addition and subtraction had been conducted with 4- to 6-year-olds. Such young children who, like the model, lack sophisticated metacognitive knowledge, may choose strategies by means of the type of low-level process depicted in the model. However, 8- and 9-year-olds, with much more sophisticated metacognitive knowledge (Kreutzer, Leonard, & Flavell, 1975), may rely on that metacognitive knowledge to choose strategies. Therefore, their strategy choice procedures may be quite different.

Another reason for testing the model's applicability to multiplication was to show that it was not limited to the
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domain in which it was first proposed, addition and subtraction. Models often fit their original domain better than other domains in which they theoretically should apply, because of limits on their applicability unknown to the model builder. The young age at which children learn to add and subtract is one such potential limit. The particular type of backup strategies used in addition and subtraction is another. Thus, it seemed worthwhile to examine whether the model could contribute to understanding of children's multiplication.

Previous studies have documented a number of characteristics of children's multiplication. Third, fourth, and fifth graders use diverse strategies to multiply, among them repeated addition, reference to related problems, and retrieval (Brownell & Carper, 1943). With age and experience, retrieval becomes increasingly common, and backup strategies become less common (Brownell & Carper, 1943). Problem difficulty increases with problem size, that is, with the size of the multiplicands and the product (Campbell & Graham, 1985; Murray, 1939; Norem, 1928). In general, a problem's product is the best predictor of its relative difficulty, though in some studies the size of the smaller or larger multicanlnd has been.

Two exceptions to the general pattern of larger problems being more difficult are that problems with 5 as a multiplicand and tie problems (e.g., 6 x 6) are easier than their products' sizes would suggest (Campbell & Graham, 1985). Two common types of errors that children make are multiplicant-related errors, in which the incorrect answer is a multiple of at least one multiplicand (6 x 4 = 30), and close misses, in which the incorrect answer is close to the correct product (6 x 4 = 23). One goal of Experiment 1 was to add to this set of phenomena new information about problem difficulty, particular errors, and especially characteristics of backup strategies.

The second goal of the experiment was to test two basic predictions of the distribution of associations model. Both concerned when children use backup strategies. One prediction was that backup strategies would be used most often on the same problems that have high percentages of errors and long solution times. The reason is that all three aspects of performance are hypothesized to depend on the same underlying variable: the distribution of associations between problems and answers. Relative to flat distributions (e.g., that for 6 x 9 in Figure 1), peaked distributions (e.g., that for 3 x 5) result in (a) less frequent use of backup strategies (because the answer that will be retrieved on 80% of retrieval efforts, 25, has sufficiently high associative strength to exceed many confidence criteria that no answer to 6 x 9 can exceed, thus leading to use of retrieval rather than a backup strategy); (b) fewer errors (because of the higher probability of retrieving and stating the answer that forms the peak of the distribution, which almost always, as in the Figure 1 example, is the correct answer); and (c) shorter solution times (because the probability of retrieving on an early retrieval attempt an answer whose strength exceeds any given confidence criterion is greater the more peaked the distribution). Thus, percent use of backup strategies, percent errors, and mean solution times covary because all are functions of the peakedness of the distributions of associations.

This first prediction might be expected on general grounds of difficulty; more difficult problems elicit more errors, longer solution times, and greater use of backup strategies (though the model indicates exactly why some problems are more difficult than others and specifies a mechanism that would produce the hypothesized pattern of performance). The second prediction, however, is quite specific to the present model: The primary source of the positive correlations among percentage of backup strategy use, percentage of errors, and length of solution times on each problem should be parallels between frequency of backup strategy use on each problem and percentage of errors and length of solution times on retrieval trials on the problem. The prediction follows from the reasoning in the last paragraph. Percentage of backup strategy use on a problem, percentage of errors on retrieval trials on that problem, and length of solution times on retrieval trials on the problem all are hypothesized to depend entirely on the peakedness of the problem's distribution of associations. Because all three depend on the same underlying variable, they should correlate highly. This should be true even when the common influence of problem size is partialled out, because other factors that influence the peakedness of distributions (e.g., problem presentation frequency) would contribute additional shared variance.

In contrast, percent errors and length of solution times on backup strategy trials do not depend on the peakedness of the distribution of associations. They depend on the difficulty of executing the particular backup strategy. For example, in repeated addition, percent errors and length of solution times should reflect the number of addition operations and the sizes of the numbers being added. That is, they should reflect the product. Because percent errors and length of solution times on backup strategy trials on each problem do not depend on the peakedness of the distribution of associations, they should correlate less highly with percent use of backup strategies on each problem than should percent errors and length of solution times on retrieval trials on the problem. Moreover, partialling out the common relation to product size should reduce the relation between frequency of use of backup strategies and errors and solution times on backup strategy trials to a greater degree (because the shared relation to product size is hypothesized to be the only source of this relation but not the only source of the relation between the frequency of use of backup strategies and the errors and solution times on retrieval trials).

This prediction is based on a different conception of problem difficulty from that underlying alternative models of arithmetic and most psychological models in all areas. The many psychological models that attempt to predict relative solution times and error rates on different problems rarely distinguish among alternative strategies that may have generated subsets of the data. Implicitly, problem difficulty is treated as an unconditional property of the problem that can be predicted from other characteristics of the problem (e.g., the size of the product, the number of degrees the figure is rotated; the difference in length, height, or weight of the objects whose sizes are being compared, etc.). Within the present model, however, problem difficulty is viewed as dependent on problem characteristics as such but rather on how problem characteristics influence ability to execute the alternative strategies that are used on the problem. This perspective led to the prediction that problem difficulty on
retrieval trials would differ from that on backup strategy trials in a way that would lead to percentage use of backup strategies correlating more highly with the difficulty pattern on retrieval trials on the problem than with the difficulty pattern on backup strategy trials on the problem.

Method

Participants. The children were 26 third graders, 15 girls and 11 boys, with mean chronological age of 102 months. All were from the higher of the two math achievement groups within their grade. Students in this group had been exposed to multiplication for the first time late in the second grade. At the time of testing, they had had roughly 5 months' experience with the earliest introduced problems (not counting summer vacation) and 3 weeks' experience with the most recently introduced ones. The experimenter was a 33-year-old female research assistant.

Problems. Problems were the 100 combinations of multiplicand (0-9) x multiplier (0-9). They were ordered randomly, subject to the constraint that one of each pair of inverse problems (e.g., 3 x 7 and 7 x 3) be in the first 50 problems and the other in the second 50. Half of the children received the problems in one order and the others in the opposite order. The two sets were presented on separate days within a 1-week period.

Procedure. Each child was brought individually from the classroom to a vacant room within the school. The child was seated at a table directly across from the experimenter. On the child's side of the table were 10 sheets of blank paper, stacked in a pile, with a pencil atop the first sheet. Before each session, the child was told:

We are going to do some multiplication problems today. I'll read the problems and when you have the answer, tell me what it is. You can do anything you want to get the right answer. There are several pieces of blank paper in front of you on the desk. You can use them in any way you want to get the right answer.

Then the problems were presented orally in the form “How much is \( n \) times \( m \)?”

Each child's behavior was recorded by using a Sony SLO-323 videocassette recorder and a Sony 3260 camera. Solution times were recorded through use of a Vicon X240 digitizer that printed digital times, accurate to 1/100 of a second, across the bottom of the taped scene. The experimenter also made notes about the child's behavior on each problem.

Two raters reviewed the videotapes and notes and independently classified the strategy used on each trial. Use of backup strategies was inferred from overt behavior. On trials in which no overt behavior was evident, children were classified as having retrieved the answer. This classification procedure had the potential weakness that if children executed backup strategies covertly, they would be misclassified as having used retrieval. However, such misclassifications of backup strategies as retrieval could only blur the difference between performance on retrieval and backup strategy trials and thus work against the present prediction that accuracy and solution times on retrieval trials on each problem would correlate more highly with percentage backup strategy use on that problem than would accuracy and solution times on backup strategy trials on the problem.

The raters' initial strategy classifications agreed on 96% of trials. On the other 4%, they discussed the classifications until they agreed.

Results and Discussion

Strategy use. As shown in Table 1, children appeared to use retrieval and three backup strategies: repeated addition, counting sets of objects, and writing the problem. Repeated addition involved adding one multiplicand the number of times indicated by the other. Children were more likely than chance to execute this strategy by adding the first multiplicand the number of times indicated by the second; they did so on 63% of repeated-addition trials, \( x^2(1, N = 26) = 16.45, p < .01 \). In contrast, they were exactly as likely to add the larger multiplicand the number of times indicated by the smaller as to do the reverse, proceeding in each way on 50% of trials. The counting-sets-of-objects strategy involved writing groups of tally marks on a piece of paper (e.g., 3 groups of 4 for 3 \( \times \) 4) and then counting the tally marks. Writing-the-problem involved writing the two multiplicands and answering orally without further overt behavior. Retrieval involved no audible or visible behavior between presentation of the problem and statement of the answer.

The diverse strategy use reflected individual as well as group-level behavior. Of the children, 92% used two or more strategies, 65% used three or more, and 23% used all four. Median solution times on retrieval trials were significantly faster than times on writing-the-problem trials, which were significantly faster than times produced by the other two strategies (dependent-measures \( t \)s > 5.00, \( p < .01 \)). Retrieval was also significantly more accurate than any of the other three strategies (\( t \)s > 3.00, \( p < .01 \)).

As predicted by the model, children used backup strategies most often on the same problems that elicited the highest percentage errors (\( r = .83 \)) and the longest median solution times (\( r = .86 \)). Also as predicted, the relation among percentage backup strategy use, percentage errors, and length of solution times on each problem stemmed primarily from the pattern of errors and solution times on retrieval trials on the problem. Percentage errors on retrieval trials on each problem correlated \( r = .82 \) with percentage backup strategy use on the problem. Percentage errors on repeated-addition and counting-sets-of-objects trials correlated \( r = .58 \) with percentage backup strategy use. The difference between correlations was significant, \( t(62) = 3.57, p < .01 \).1

1 As in previous studies (e.g., Siegler & Shrager, 1984), only problems on which both retrieval and backup strategies were used by at least three children were included in these comparisons of correlations. A potential confounding factor in the comparisons of correlations was that children used retrieval more often than the repeated-addition and counting-sets-of-objects strategies. On the 63 problems over which the correlations were computed (the 63 problems on which children used both retrieval and the two backup strategies at least three times), children used retrieval on 56% of trials and the backup strategies on 42% of trials. To overcome this potential con-
An even more specific prediction of the model was that the correlations would differ even when the product was partialed from both of them. This prediction also proved correct. When the product was partialed, percentage backup strategy use on each problem continued to correlate significantly with percentage errors on retrieval trials on the problem \( r = .43, p < .01 \). No significant correlation remained, however, when the product was partialed from the correlation between percentage use of backup strategies and percentage errors on repeated-addition and counting-sets-of-objects trials (the two strategies in which problem difficulty was expected to depend on product size), \( r = .11, p > .10 \). The fact that a significant correlation between percentage use of backup strategies and percentage errors on retrieval trials remained when the product was partialed out demonstrated that this correlation was not due just to children using backup strategies on larger problems.

Correlations between solution times and backup strategy use also were in accord with the model's predictions, at least when the product was partialed out. The correlation between percentage backup strategy use on each problem and median solution times on retrieval trials on it was actually nonsignificantly lower than that between percentage backup strategy use on each problem and median solution times on repeated-addition and counting-sets-of-objects trials on it \( r = .72 \) vs. \( r = .78, t < 1 \). When the product was partialed from the two correlations, however, the picture changed dramatically. The correlation between percentage backup strategy use on each problem and median solution time on retrieval trials remained significant \( r = .58, p < .01 \). In contrast, the correlation between percentage backup strategy use on each problem and median solution time on repeated-addition and counting-sets-of-objects trials plummeted to almost zero \( r = .02, p > .10 \). Thus, as predicted by the model, the correlation between percentage backup strategy use and solution times on repeated-addition and counting-sets-of-objects trials was based entirely on the common relation to product size, whereas the correlation between percentage backup strategy use and length of solution times on retrieval trials on each problem reflected additional shared influences.

**Problem difficulty**. Beyond testing the model's predictions concerning when backup strategies are used, a second purpose of Experiment 1 was to identify factors that influence problem difficulty and particular errors in children's multiplication. To examine problem difficulty, stepwise regression analyses were run on the predictors of each problem's frequency of errors and median solution times. The predictors examined were the sizes of the product, the sum, the first multiplicand, the second multiplicand, the smaller multiplicand, and the larger multiplicand; whether the problem included a 5; and whether the problem was a tie.

In the analysis of solution times, the first three variables to enter the regression equation were the size of the product, whether the problem included 5 as a multiplicand, and whether the problem was a tie. Each added significant independent variance to that which could be accounted for by the other two. The size of the product accounted for 83% of the variance in median solution time on each problem, whether the problem included 5 as a multiplicand brought the variance accounted for to 85%, and whether the problem was a tie brought the variance accounted for to 86%.

In the analysis of percentage of errors, the product and whether the problem included 5 as a multiplicand were the first two variables to enter the regression equation. Each added significant independent variance; the product accounted for 79%, and whether the problem included a 5 raised the variance accounted for to 80%. Whether the problem was a tie did not add significant independent variance in this analysis.

**Particular errors**. Slightly more than half of the errors (51%) were multiples of one or both multiplicands. The frequency of such errors exceeded the frequency expected by chance on 83 of the 96 problems for which chance probabilities could be calculated \( p < .01 \), Wilcoxon test.

Close-miss errors, defined as errors within 10% of the product, also were common. Such errors were possible only on the 58 problems with products of 10 or more; on problems with smaller products, no incorrect answer was within 10% of the product. Across the 58 problems on which close-miss errors were possible, 29% of the errors were close misses. To compute the chance likelihood that an error would be this close in magnitude on a particular problem, the percentage of answers that were within 10% of that problem's product (but not identical to it) was computed for all problems except the original one. Comparing frequencies of observed and expected close misses indicated that children produced more close misses than expected by chance on 53 of the 58 problems \( p < .01 \), Wilcoxon test.

A third, common type of error, related-operation errors, had previously been reported for adults (Miller et al., 1984; Winkelman & Schmidt, 1974) but not for children. These were errors that would have been correct for addition (e.g., 4 \( \times \) 3 = 7). Across the 100 problems, 22% of the errors were of this type. Such related-operation errors were especially common on problems with 0 as a multiplicand (99% of errors on these problems) but were also fairly common on other problems (13% of errors).

A final regularity in children's errors involved their odd-even status. Krueger (1986) found that on a verification task, adults took longer and were less accurate in rejecting errors that had the correct odd-even status for the pair of multipliers.

---

2 The formula used to compute for each problem the chance probability that an error was a multiple of a multiplicand was \( 1/N + 1/M - 1/NM \), where \( N \) and \( M \) correspond to the two multiplicands in the problem. Problems involving 0s and 1s as multiplicands represented special cases for this formula, because all errors would necessarily be multiples of 1 and because no error could be a multiple of 0 when 0 was a multiplicand. On problems in which one multiplicand was 0 or 1, the chance probability was defined as \( 1/N \), where \( N \) was the other multiplicand. On the four problems in which both multiplicands were 0 or 1, no chance probability could be computed.

---

founding, 24% of retrieval trials were randomly deleted from the data for each problem, leaving equal numbers of retrieval and backup strategy trials, and the correlation between percentage backup strategy use and percentage errors on retrieval trials was recomputed. As in the original analysis, percentage backup strategy use on each problem correlated more highly with percentage errors on retrieval trials than with percentage errors on the backup strategy trials \( r = .76 vs. r = .58 \), \( t(60) = 2.31, p < .05 \).
cands (e.g., two odd multiplicands always produce an odd product) than the answers' distance from the correct product would predict. The odd-even status of children's errors on the present production task did not reflect this pattern; it conformed to it on exactly 50% of trials. The children's errors did show an interesting variant of the pattern, though. On 65% of trials, their answers conformed to the odd-even status that would have followed from adding the multiplicands. When multiplying two odd numbers, children's errors were usually even (57%); when multiplying an odd and an even number, their errors were usually odd (68%); when multiplying two even numbers, their answers were usually even (80%). The deviation from chance was highly significant, \( \chi^2(1, N = 26) = 64.36; p < .01 \). The phenomenon seemed to be another indicator that knowledge of addition and multiplication is interconnected even (perhaps especially) early in the acquisition of multiplication.

The present and previous findings about problem difficulty, particular errors, and strategy use constitute a substantial body of phenomena for a model to explain. Table 2 summarizes these findings and indicates the hypothesized main source of each phenomenon within the present model. Some phenomena are attributed to relative difficulty of executing backup strategies on different problems; others are ascribed to relative frequency of presentation of problems; yet others are attributed to the workings of the retrieval mechanism. In the next three sections, the role of each factor is tested empirically.

Experiment 2: Repeated Addition Performance

The model suggested that four phenomena are due to the workings of backup strategies: the problem size effect, the multiples-of-5 effect, multiplicant-related errors, and close-miss errors. The problem size effect is hypothesized to arise because the larger the numbers being added or counted and the more numbers that need to be added or counted, the more often children will err in executing the repeated-addition and counting-sets-of-objects strategies. The 5s effect would occur because children add 5s more accurately than other numbers of comparable size, such as 4s and 6s (Ashcraft, Fierman, & Bartolotta, 1984). Multiplicant-related errors would follow from children executing the repeated-addition strategy correctly except for writing down or adding the multiplicant too few or too many times. Close misses would occur through children counting and adding in a way that produces errors close in magnitude to the correct answer. Experiment 2 was designed to determine whether children's use of the main multiplication backup strategy, repeated addition, in fact produces answers that have these properties and thus could account for these four phenomena.

Method

Participants. Children were 22 fourth graders, 12 boys and 10 girls, from the same school as the children in Experiment 1. All were in the lower of the two math achievement groups within their grade. Their mean chronological age was 116 months; none had participated in the earlier experiment. The experimenter was the same research assistant as in Experiment 1.

Problems. Each item involved a column of numbers that corresponded to a single-digit multiplication problem. For each problem, the first multiplicant was written the number of times indicated by the second. For example, 8 × 6 became a column in which 8 was written 6 times. Literal multiplication problems were not present on the sheet; only columns of numbers were. Among the 100 single-digit problems, 28 could not be presented in this way; they included either 0 as one of the two multiplicands or 1 as the multiplier. It would have been literally impossible to present in this format problems in which the multiplier was 0 and, at minimum, would have been odd to ask children to sum a single number or a row of 0s. Therefore, children were presented only the 72 other problems.

Procedure. Children were brought individually to the same room as in Experiment 1. They were seated at a desk across from the experimenter and given sheets of paper that had typed on them the columns of numbers corresponding to each multiplication problem. Children also were provided scratch paper. The problems were presented on 2 successive days, with half of the problems presented each day. The instructions were "Today I want you to add up these columns of numbers and write down the answer when you are done."
Results and Discussion

Relative problem difficulty. A stepwise regression analysis of the percentage correct on each problem was conducted by using the same predictors as in Experiment 1. The two significant predictors were product size, which accounted for 73% of the variance, and whether the number being added was a 5, which raised the variance accounted for to 80%. Thus, both of the effects on problem difficulty that were attributed to use of backup strategies were evident in the answers generated by repeated addition.

The nature of the effect for adding 5s provided evidence that children followed the instructions to add the numbers rather than multiplying. If they had multiplied, the ease of problems with 5 as a multiplicand would be equally apparent when adding a column of 5s and when adding some other number five times. On the other hand, if they followed instructions to add the numbers, the ease should be apparent only on problems in which 5s were added. As shown in Table 3, children erred in adding 5s on only 3% of trials, whereas they erred in adding other numbers 5 times on 23% of trials.

Particular errors. Repeated addition also often generated both types of errors attributed to it: multiplicand-related errors (37% of total errors) and close misses (48%). With the probabilities calculated as in Experiment 1, multiplicand-related errors occurred more often than chance on 39 of the 48 problems on which children erred at least once, and close misses occurred more often than chance on 41 of the 48 (ps < .01, Wilcoxon test).

In sum, repeated addition produced answers that, if associated with the problems on which they were stated, would generate all four effects attributed to relative difficulty of executing backup strategies.

Problem Presentation Frequency

Only one phenomenon was hypothesized to derive primarily from relative frequency of problem presentation: Tie problems are easier than their numerical sizes would predict. To test whether multiplication ties are in fact presented more frequently than other problems of similar size, two second-grade textbooks were examined. The Addison-Wesley Mathematics Series Workbook (Eicholz et al., 1983) was examined because it was used by the children who partici-ipated in these experiments. The Heath Elementary Mathematics Series Workbook (Dilley, Rucker, & Jackson, 1975) was examined to test the generality of the Addison-Wesley pattern.

In a dependent-measures t test, the number of presentations of each tie problem was compared with the number of presentations of the average of the 18 nontie problems containing that multiplicand. Thus, the number of times 4 × 4 was presented was compared with the average presentation frequency for the other 18 problems that included 4 as a multiplicand. As shown in Table 4, in the Addison-Wesley series, tie problems were presented 1.47 times as often as nontie problems that shared a multiplicand, t(9) = 2.84, p < .05. A parallel analysis of the Heath series workbook showed that in it, ties were presented 1.3 times as often as nontie problems, t(9) = 3.75, p < .01. The finding suggests that presentation frequency contributes to the relative ease of tie problems in multiplication, as had previously been found with addition.

Experiment 3: Retrieval Performance

In this experiment, performance was observed under conditions requiring use of retrieval on all trials. One reason that performance under retrieval-required conditions was of interest involved the model's predictions about factors that influence retrieval. Within the model, whatever factors influence the formation of the distributions of associations should also influence performance on retrieval trials. The three factors hypothesized to influence formation of the distribution of associations are backup strategies, problem presentation frequencies, and connections to other numerical operations. Therefore, the effects attributed to these three factors should all be apparent in retrieval performance. These include the four effects attributed to backup strategies (the problem size effect, the 5s effect, multiplicand-related errors, and close-miss errors), the one effect attributed to presentation frequencies (the ties effect), and the two effects attributed to connections to addition (related-operation errors, such as 4 × 3 = 7, and the odd-even effect).

A second reason why performance under these retrieval-required conditions was of interest was that it provided a direct measure of the associative strengths within children's multiplication networks. In studying addition, Siegler and Shrager (1984) used the proportion of trials on which children stated a given answer to a given problem under such retrieval-required conditions to estimate the associative strength linking the answer to that problem. The estimates proved useful both as a target against which to evaluate the adequacy of the Siegler and Shrager (1984) computer simulation and also as the starting point of the associative strengths within Ashcraft's (1987) simulation of older children's addition. As will be described later, the corresponding data for multiplication were put to both similar and new uses in the present study.

Method

Participants. The children were 27 third graders and 20 fourth graders (mean chronological age = 108 months), 26 girls and 21 boys, who attended the same school as did children in the previous exper-
Table 4
Number of Times Each Problem Presented in Second- and Third-Grade Workbooks (Addison-Wesley Series)

<table>
<thead>
<tr>
<th>Multiplicand (first number)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>19</td>
<td>18</td>
<td>17</td>
<td>20</td>
<td>18</td>
<td>17</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>18</td>
<td>24</td>
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<td>19</td>
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<td>15</td>
<td>15</td>
<td>13</td>
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<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>19</td>
<td>20</td>
<td>19</td>
<td>20</td>
<td>14</td>
<td>11</td>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>9</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>13</td>
<td>14</td>
<td>11</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>12</td>
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<td>6</td>
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<td>9</td>
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<tr>
<td>8</td>
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<td>3</td>
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<td>9</td>
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<td>1</td>
<td>5</td>
<td>11</td>
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<td>7</td>
<td>8</td>
<td>9</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

...expected by chance on 50 of the 58 problems for which chance probabilities could be calculated (p < .01, Wilcoxon test). Related-operation errors also were quite frequent (17% of errors). Finally, as in Experiment 1, more errors (60%) were in accord with the odd–even status produced by adding the two multiplicands than would have been expected by chance, $\chi^2(1, N = 47) = 52.54, p < .01$; and the percentage in accord with the odd–even status produced by multiplying the multiplicands, 54%, did not differ from chance, $\chi^2(1, N = 47) = 1.89, p > .05$.

In sum, when children were instructed to use retrieval on all trials, six of the seven phenomena that were expected to emerge were evident. These included two of the three phenomena involving relative problem difficulty and all four involving particular errors.

Predictions About Development

This section examines a basic developmental prediction of the model: Third and fourth graders' percentage correct on retrieval trials should predict older children's and even adults' solution times on that problem better than any structural variable, such as the product or the sum of the numbers. The logic underlying this prediction is straightforward. Within the present model, both accuracy and solution times on retrieval trials depend entirely on the distribution of associations. Early-emerging differences in relative peakedness, and therefore in relative error rates and solution times, should be maintained unless one of the factors contributing to development alters the initial pattern. The analysis described below did not indicate any factors that would alter the pattern; therefore the predicted parallels, despite the influence of 10 years of experience with multiplication.

The analysis underlying this prediction involved examining potential changes in each of the three factors hypothesized to shape development of the distribution of associations. First, consider backup strategies. These can be divided into those that third and fourth graders already know and those that they may acquire later. The main backup strategy that third and fourth graders already know is repeated addition. With experience, addition becomes faster and more accurate on all problems. However, relative problem difficulty should be maintained. There is no reason to expect that ease of adding larger numbers ever exceeds that of adding smaller numbers or that ease of adding more numbers ever exceeds that of adding fewer numbers. Similarly, counting by 5s seems to continue to be easier than counting by any other number except 1 and 2, even for adults. Thus, these backup strategies would not be expected to change the early-developed patterns of problem difficulty.

Now consider the potential role of later acquired backup strategies. The model suggests that relative problem difficulty will change if children acquire and frequently use new backup strategies that produce different patterns of difficulty. For example, in single-digit addition, children's predominant backup strategy changes from counting from one to counting from the larger number. When using the earlier strategy, 12 + 3 is more difficult than 7 + 5; when using the later strategy, the opposite is the case. No similar shifts in predominant
multiplication backup strategies have been documented, however. Thus, newly learned backup strategies also were not expected to change relative problem difficulty.

Next, consider potential changes over time in the way that related numerical operations affect relative problem difficulty. The experiments reported in this article indicate that third and fourth graders' associative networks already include strong links between multiplication and addition. New links between multiplication and division are established later in schooling. However, these links seem likely to reinforce the existing pattern of relative problem difficulty rather than changing it. Either repeated addition of the divisor or reference to related multiplication problems can be used as a backup strategy for division; both of these tend to make relative difficulty of division problems resemble relative difficulty of corresponding multiplication problems. Consistent with this view, Campbell (1985) found that the best predictor of adults' solution times on division problems was their solution times on corresponding multiplication problems. Thus, there was little reason to think that new links across numerical operations would alter relative problem difficulty.

Finally, consider the third hypothesized source of change, presentation rates. Without studying the problems presented in mathematics classes, science classes, and everyday life, it is impossible to make firm predictions about this factor. It seems likely, though, that because of the importance of squares in solving mathematics and science problems, older students, like younger ones, receive more exposure to multiplication ties than to other problems. Furthermore, there was no obvious reason to expect other subgroups of problems to be presented unusually often and thus to undercut the initial pattern.

Taken together, these hypotheses about the course of development suggest the following prediction: if (a) third and fourth graders' percentage correct in the retrieval-required experiment provides a good index of the relative peakedness of their distributions of associations on each problem, (b) the relative peakedness of the distributions is preserved over age, and (c) the peakedness of the distribution of associations determines both percentage errors and length of solution times on retrieval trials, then third and fourth graders' percentage correct in the retrieval-required experiment should accurately predict the solution times of older children and even adults.

The prediction can be taken a step further. Third and fourth graders' percentage correct in the retrieval-required experiment should be a better predictor of older children's and adults' solution times than any single structural variable, such as the size of the product or sum of the multiplicands. The reason is that the distribution of associations responds to the multiple factors that shape both children's and adults' responding, whereas each structural variable corresponds to only one of these factors. For example, a problem's product is a good index of the relative difficulty of executing backup strategies on that problem. Unlike the distribution of associations, however, the product does not reflect the influence of problem presentation frequencies or whether 5 is a multiplier.

This prediction differs from that of alternative models. Geary and Widaman (1987) and Geary, Widaman, and Little (1986) hypothesized that product size would be the best predictor of adults' solution times. Their argument was that adults' multiplication knowledge is arranged in a tabular format, that activation spreads outward from 0 x 0, and that solution time is proportional to the area that must be activated, that is, to the product. Stazyk et al.'s (1982) model makes similar assumptions, except that because activation spreads from the two multiplicands, the sum of the multiplicands should best predict solution times. The present model predicts that the sum and especially the product of the multiplicands should correlate positively with solution times, due to their relation to the ease of using backup strategies, but that the peakedness of young children's distributions of associations should correlate more highly with adults' performance.

To test this prediction, I reanalyzed results of all previous studies of single-digit multiplication that used production tasks and that reported mean or median solution times for each problem. Four articles provided such data: Jerman (1970), Campbell and Graham (1985; adult subjects only), Aiken and Williams (1973), and Miller et al. (1984). Three analyses were performed on each data set. First, a stepwise regression analysis was used to determine the best predictor of each problem's mean solution time. This analysis included the structural predictors used in the previous regression analyses in this study plus the third and fourth graders' percentage correct on the problem in the retrieval-required experiment (Experiment 3). This last variable is referred to as the associative strength variable, because it is the operational definition of the percentage of associative strength concentrated in the correct answer on the problem.

The other two analyses compared the predictive power of the associative strength variable with that of the best structural predictor. In one, the best structural predictor was defined as the best structural predictor within the individual data set. In the other, the best structural predictor was defined as the best predictor across all data sets, the product. (This last analysis was conducted only when the best structural predictor for the particular data set was not the product.) In these analyses, the associative strength variable and the best structural predictor were forced into a regression equation in the two possible orders, and the independent variances contributed by the two were compared.

First, consider the reanalyses of Jerman's (1970) data on third through sixth graders' solution times. As shown in Table 5, at all ages the associative strength variable accounted for at least 10% more variance than did any structural variable. Furthermore, the associative strength variable always added significant independent variance to that which could be explained by the best fitting structural variable, whereas the best fitting structural variable added significant independent variance at only one of the four ages.

Now consider the reanalyses of adults' solution times. In Aiken and Williams' (1973) data set, the associative strength variable accounted for 10% more variance than the best fitting structural variable, the product. In Miller et al.'s (1984) data set, the associative strength variable and the best fitting structural variable, the product, were equally good predictors of solution times. In Campbell and Graham's (1985) data set, the associative strength variable predicted as well as the best fitting structural predictor in that experiment, the larger num-
Table 5
Percentage Variance in Solution Times Accounted for by Associative Strength Variable and by Best Structural Predictor

<table>
<thead>
<tr>
<th>Data set</th>
<th>Associative strength variable entered first</th>
<th>Best structural predictor entered first</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Predictor</td>
<td>$R^2$ increment</td>
</tr>
<tr>
<td>Jerman (1970)</td>
<td>1. AS</td>
<td>85</td>
</tr>
<tr>
<td>Third graders</td>
<td>2. AS</td>
<td>1</td>
</tr>
<tr>
<td>Jerman (1970)</td>
<td>1. AS</td>
<td>78</td>
</tr>
<tr>
<td>Fourth graders</td>
<td>2. Small</td>
<td>0</td>
</tr>
<tr>
<td>Jerman (1970)</td>
<td>1. AS</td>
<td>78</td>
</tr>
<tr>
<td>Fourth graders</td>
<td>2. Product</td>
<td>0</td>
</tr>
<tr>
<td>Jerman (1970)</td>
<td>1. AS</td>
<td>81</td>
</tr>
<tr>
<td>Fifth graders</td>
<td>2. Product</td>
<td>1</td>
</tr>
<tr>
<td>Jerman (1970)</td>
<td>1. AS</td>
<td>74</td>
</tr>
<tr>
<td>Sixth graders</td>
<td>2. Product</td>
<td>2*</td>
</tr>
<tr>
<td>Aiken &amp; Williams</td>
<td>1. AS</td>
<td>72</td>
</tr>
<tr>
<td>(1973) adults</td>
<td>2. Product</td>
<td>2*</td>
</tr>
<tr>
<td>Campbell &amp; Graham</td>
<td>1. AS</td>
<td>52</td>
</tr>
<tr>
<td>(1985) adults</td>
<td>2. Large</td>
<td>10*</td>
</tr>
<tr>
<td>Campbell &amp; Graham</td>
<td>1. AS</td>
<td>52</td>
</tr>
<tr>
<td>(1985) adults</td>
<td>2. Product</td>
<td>0</td>
</tr>
<tr>
<td>Miller, Perlmutter, &amp; Keating</td>
<td>1. AS</td>
<td>56</td>
</tr>
<tr>
<td>(1984) adults</td>
<td>2. Product</td>
<td>4*</td>
</tr>
</tbody>
</table>

Note: AS = associative strength variable; Small = smaller multiplicand; Large = larger multiplicand. In cases in which best structural predictor was not product, best structural predictor was used in upper comparison and product in lower. For an illustration of the meaning of the numbers in this table, consider the topmost example, Jerman's (1970) data on third graders. When the associative strength variable was entered first into the regression equation, it accounted for 85% of the variance in solution times on each problem, and the product added 1% to the variance that could be explained. When the best structural predictor, the product, was entered first, it accounted for 73% of the variance, and the associative strength variable added 13%.

* Contribution of second variable to enter regression equation significant, $p < .01$.

ber, and accounted for 15% more variance than did the product. In all cases, the associative strength variable added significant independent variance to that which could be accounted for by the best fitting structural variable. Thus, as suggested by the model, the associative strength variable, based on third and fourth graders' percentage correct retrieval, predicted adults' solution times better than any structural variable.

A Simulation of Multiplication Learning

To test the sufficiency of the model to simultaneously account for the diverge aspects of performance and learning documented in these experiments, a computer simulation was written. It proved able to produce both immediate performance and learning much like those of children.

The Simulation's Operating Procedure

The simulation's initial state was intended to model children's knowledge at a point when they had received premultiplication instruction but not experience with multiplication problems. The initial representation included only minimal associations between each multiplication problem and each whole number between 0 and 100 (associative strengths = .01). The process included the Figure 2 retrieval mechanism and the backup strategy of repeated addition.

The way in which the simulation progressed beyond this initial state can be summarized in terms of seven features of its operation:

1. Each problem was presented in accord with that problem's relative frequency in the workbook used by children in these experiments.
2. Before each item, the simulation generated a confidence criterion and a search length. The values were generated randomly and independently on each trial from among the 90 confidence criterion values from .05 to .95 and from among the three search length values 1, 2, and 3.
3. Next, the simulation retrieved an answer. The probability of retrieving any given answer was proportional to its associative strength relative to the total associative strengths of all answers to the problem. If the associative strength of the retrieved answer exceeded the current confidence criterion, that answer was stated. Otherwise, retrieval attempts continued until such an answer was retrieved or until the number of searches matched the allowed search length, whichever came first.
4. On a randomly chosen 20% of trials on which no answer had been advanced before the maximum search length was reached, one last answer was retrieved and stated (the sophisticated guessing approach). The probability of a given answer being retrieved was proportional to its associative strength relative to those of all answers connected to that problem.
5. If no answer had been stated, the program proceeded in a way intended to model the effects of writing the problem. The associative strength of each answer was boosted by 5% of its current problem–answer associative strength. This increment did not change the probability of a given answer being retrieved but did make all answers more likely to be
stated if retrieved. Then, the simulation retrieved an answer as described earlier and stated it if its associative strength exceeded the confidence criterion.

6. If no answer had been stated, the simulation represented one multiplicand the number of times indicated by the other and summed the result. On one half of trials, the multiplicand was represented the number of times indicated by the multiplier; on the other one half, the reverse was done. Errors could arise in two ways. First, on each addition operation, there was a fixed probability of adding the same number twice or of not adding a number at all. Second, the simulation could add a number incorrectly. For example, on $2 \times 8$, it might add $8 + 8$ and answer $15$. Frequency of such errors increased linearly with the size of the number being added, except for additions of $5$s, which were performed as accurately as additions of $2$s. Particular errors were normally distributed around the correct sum.

7. Every time the system advanced an answer, the association between that answer and the problem increased. The increment was twice as great for correct answers—which presumably would often be reinforced—as for incorrect answers, which presumably would rarely be.

The simulation ran in two phases: a learning phase and a test phase. The learning phase was designed to resemble children's experience with multiplication prior to the experiment. The test phase was designed to resemble their experience in the experimental setting. Operationally, the phases differed in two ways. One concerned presentation rates. Presumably, before the experiment, children encountered some problems more often than others, whereas during it, all problems were presented equally often. Therefore, in the learning phase, presentation rates were based on those in the Addison-Wesley workbook, whereas in the test phase, all problems were presented equally often. The other difference concerned changes in associative strengths. During the learning phase, intended to model each child's preexperimental experience, the association between problem and answer increased whenever the simulation answered a problem. During the test phase, intended to model different children's performance in a situation in which each child encountered each problem only once, associative strengths stayed constant.

The size of the associative strength matrices created on each problem led to the simulation running very slowly on the VAX-780 that I used. Therefore, it was run on only 20 of the 100 multiplication problems: $0 \times 2, 0 \times 6, 1 \times 4, 1 \times 7, 2 \times 1, 2 \times 5, 3 \times 3, 3 \times 9, 4 \times 0, 4 \times 8, 5 \times 2, 5 \times 6, 6 \times 1, 6 \times 5, 7 \times 3, 7 \times 7, 8 \times 4, 8 \times 9, 9 \times 0, 9 \times 8$. This subset was chosen for being as representative of the total set as possible. Each of the 10 digits was used as the multiplicand on two problems and as the multiplier on two problems. Percentage of tie problems in the sample, 10%, was identical to that in the parent population. Mean product size, 20, also was identical for the subset and the parent set.

The Simulation's Behavior

The simulation's test phase performance paralleled that of children in numerous ways. First, consider overall problem difficulty. Percentage errors on each problem of the simulation and the children correlated $r = .90$. Solution times correlated $r = .95$. Furthermore, the same factors that influenced problem difficulty for children—product size, the tie effect, and the ties effect—also influenced the simulation. Product size correlated .79 with the simulation's percentage errors. Problems with 5 as a multiplicand were solved more often than other problems (92% vs. 87%), despite mean product size being identical for problems with and without a 5. Tie problems also were easier than would be expected from the sizes of their products. The tie problems' mean product size was larger than that of the problems as a whole (29 vs. 19). However, the percentage correct was almost as great for ties as for other problems (86% vs. 88%). Seen from another perspective, eliminating the 6 nontie problems with the lowest products resulted in the remaining 12 nontie problems having the same mean product size, 29, as the 2 tie problems. Percentage correct for these nontie problems was 81% (vs. 86% for the ties).

Particular errors generated by the simulation also resembled those generated by children. Of the simulation's errors, 40% were multiples of one of the problem's multiplicands (vs. 34% that would have been expected by chance). Furthermore, 48% of the simulation's errors were within 10% of the correct product (vs. 5% expected by chance).

Of particular interest, the simulation's strategy choices, made without any rational calculation of the advantages and disadvantages of each strategy, closely resembled the choices of the 8- and 9-year-olds in this study. The simulation's percentage backup strategy use on each problem correlated $r = .90$ with that of the children. The simulation also produced the high correlations among percentage backup strategy use, percentage errors, and solution times that characterized children's performance (Table 6).

These measures all concern performance at the end of learning; what of the learning phase itself? Again, the simulation's pattern of learning was much like that of children. From the first 200 to the last 200 trials of the learning phase, use of retrieval increased from 24% to 77% of trials. Moreover, percentage correct increased from 61% on the first 200 trials of the learning phase to 89% on the last 200. Within this context of improvement in absolute terms, relative problem difficulty showed considerable continuity. Each problem's percentage correct on the first 200 and last 200 learning trials correlated $r = .70$.

The simulation was also used to test whether the model would produce the observed close parallel between third and

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<tr>
<td><strong>Computer Simulation's Performance in Test Phase</strong></td>
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<td>Intramodal correlations</td>
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<td>$r$ errors and backup strategy use $= .97$</td>
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<tr>
<td>$r$ errors and mean solution times $= .99$</td>
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<td>$r$ backup strategy use and mean solution times $= .98$</td>
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<tr>
<td>Correlations between children's and model's behavior</td>
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<td>$r$ errors produced by model and children $= .90$</td>
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<td>$r$ backup strategy use produced by model and children $= .90$</td>
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<td>$r$ mean solution times produced by model and children $= .95$</td>
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Note. Correlations involving errors and solution times refer to percentage errors on retrieval trials and lengths of solution times on retrieval trials. Children's data are from Experiment 1.
fourth graders’ percentage correct in the retrieval-required experiment and adults’ solution times. To allow this test, the simulation was run for enough trials to reach the very low error rate characteristic of adult performance. After 4,000 learning trials (distributed over the 20 problems), it reached such a point (99.5% correct). Then the simulation’s mean solution time on each problem was correlated with the 8- and 9-year-olds’ percentage correct in the retrieval-required experiment. The correlation was \( r = .94 \). Thus, it was not just a verbal argument that the model predicted the parallels between 8- and 9-year-olds’ distributions of associations and adults’ solution times. The same simulation that initially produced childlike performance later generated error rates and solution times much like adults’.

General Discussion

This study’s main purposes were to present a model of strategy choice and to demonstrate the model’s applicability to children’s multiplication. The model proved able to account for a wide range of phenomena concerning problem difficulty, particular errors, strategy choices, and changes over time. The findings indicate that even at 8 and 9 years, ages at which children possess extensive metacognitive knowledge, many of their strategy choices may fall out from efforts to retrieve answers, rather than through reference to that metacognitive knowledge. In this concluding section, I examine the model’s implications for an understanding of arithmetic, of cognitive self-regulation, and of cognitive development.

Implications for Arithmetic

One implication of the model concerns how children and adults represent arithmetic knowledge. A number of investigators have suggested that children and adults represent arithmetic knowledge in a tabular form, much like the tables in arithmetic books (e.g., Ashcraft, 1982; Campbell & Graham, 1985; Geary & Widaman, 1987; Kaye, Post, Hall, & Dineen, 1986; Miller et al., 1984). The central piece of evidence for this position is that the sizes of the squared sum and the product are often the best predictors of performance in addition and multiplication, respectively. As Ashcraft (1987) and Miller et al. have noted, however, inferring the form of an arithmetic representation from the best predictor of solution times is a questionable practice. High correlations among predictors often result in small and unstable differences in the variance that different predictors explain. These small differences have led to frequent disagreements about the form of the arithmetic tables, for example, whether the entries within them are spaced evenly (Geary & Widaman, 1987) or unevenly (Ashcraft, 1982). Hypothesizing such tabular representations also makes problematic the process by which people retrieve incorrect answers that are not part of any single-digit arithmetic table (e.g., \( 4 \times 7 = 29 \)). Furthermore, the view makes it unclear why factors other than problem size (e.g., the presence of 5 as a multiplicand) should influence solution times or error rates.

To deal with these problems, Ashcraft (1987) and Miller et al. (1984) suggested that both accessibility of answers and the tabular form of the representation influence arithmetic performance. This proposal increases the flexibility of the models. However, it also renders them untestable, because any aspect of performance not attributable to the hypothesized organization can be attributed to differences in accessibility.

The present model suggests a different approach to representing associative knowledge of arithmetic. From this perspective, problem–answer links differ in associative strength (accessibility), but there is no reason to postulate any tabellike organization. Abandoning the tabular metaphor would increase the compatibility of models of arithmetic with more general models of memory (e.g., Anderson, 1983; Gillund & Shiffrin, 1984). Additionally, nothing obvious would be lost by abandoning it. The phenomena said to be explained by it—notably problem size effects and multiplicand-related errors—can be explained in other ways. For example, the present model suggests that both effects are attributable to the answers generated by backup strategies. Regardless of whether this interpretation proves correct, it seems likely to be more fruitful to identify factors that lead to answers being differentially accessible than to continue pursuing the arithmetic table metaphor, either alone or in combination with the accessibility construct.

Implications for Self-Regulation

One of the most important features of the distribution of associations model is that it indicates in detail how a self-regulatory process could operate. The need for self-regulatory processes—executive processes, metacomponents, autonomous regulation, and so on—has been persuasively argued previously, but the way in which they accomplish their function has not been clearly elaborated. The present mechanism both resembles and differs from previous suggestions. The mechanism resembles Piagetian and Vygotskian suggestions in that the child’s own activity determines future strategy choices. It differs from these and numerous other approaches, however, in that the self-regulation does not depend on reflection or on any other separate governmental process. Instead, it is part and parcel of the system’s basic retrieval mechanism.

The model also suggests distinctions among three types of self-regulation, while at the same time illustrating how all three can be produced by a single mechanism. One type of self-regulation is the ability to choose strategies intelligently at any given time. The model displays this type of self-regulation in that it leads to the faster retrieval approach being used most often on problems in which it can produce accurate performance and to the more time-consuming backup strategies being used most often on problems in which they are necessary to produce accurate performance. A second type of self-regulation is adaptation to changing circumstances. The model produces this type of self-regulation through increasing associative knowledge being accompanied by increasing use of retrieval and decreasing use of backup strategies. The third type of self-regulation might be described as a self-righting capacity, the ability to recover from errors and initial unfavorable experience. The heavy use of backup strategies when problems are not well-known confers this type of stability on...
the model. Illustratively, within the run of the simulation of adult performance described above, the first four answers to 8 × 9 were incorrect, as were 8 of the first 10. Yet, by the end of the learning phase, 72 was being advanced as the answer on 99% of trials. The reason was that over trials, the backup strategies produced 72 more often than any other answer, which led to its associative strength increasing and therefore to its being retrieved and stated increasingly often. This ability to recover from initial unfavorable experience is a rarely discussed aspect of self-regulation but may be one of the most important characteristics of a self-governing cognitive system.

To what range of tasks might the present self-regulatory mechanism apply? A recent study (Siegler & Campbell, in press) demonstrates that its applicability extends beyond arithmetic. The relations among backup strategies, errors, and solution times that appeared in addition, subtraction, and multiplication also were found in first graders' reading (word identification). Once again, strategy use varied both within problems and within children; use of backup strategies (in this case mainly sounding out) increased with frequency of errors and lengths of solution times on the problem; the relation among backup strategy use, errors, and solution times was primarily due to the pattern of errors and solution times on retrieval trials; and the relations remained when word length was parted out. The finding is consistent with the view that the model should apply whenever people choose between retrieval and use of backup strategies.

Brown and Reeve (1986) noted that many other quite-complex-looking decision-making processes may also be produced by "mindless" cognitive activities, such as those embodied in the present model. Smith and Osherson (in press) made such an argument for clinical versus statistical decision making. Wyer and Srull (1986) for personality judgments, Hayes-Roth and Hayes-Roth (1979) for planning, LeFevre and Bisanz (1986) for series completion, Kahneman and Richards (1986) for referential communications, and Reder (1987) for plausibility judgments. In all these areas and many others, decisions concerning what to do may be produced by surprisingly low-level processes.

Where does all this leave the type of explicit, mindful regulation of cognition often discussed under such headings as metacognition? The present computer simulation's success in generating adaptive strategy choices without mindful intervention demonstrates that such intervention is not necessary for effective self-regulation. This is certainly not to say that such mindful regulation has no role. Rather, the question becomes, When is it influential? One possibility is that the explicit, mindful type of regulation is most influential early and late in the learning process and that the implicit, mindless type of regulation is most influential in the intermediate phases. Brown and Reeve (1986) suggested that mindful cognition "goes underground"; that is, it is influential when children encounter new problems but becomes less so as problems become familiar. Another possibility is that when problems become very familiar and require few cognitive resources, the mindful processes return at times. This reappearance may be when experts gain the deep understanding that often accompanies advanced procedural skill (Simon & Simon, 1978).

Implications for a General Understanding of Development

The present model provides perspectives on a number of long-standing issues concerning development, among them the nature of developmental sequences and the relation between current behavior and change over time.

Developmental sequences. Many descriptions of development hypothesize sequences of knowledge states leading to mature understanding of concepts. These hypothesized sequences typically involve a 1:1 relation between a child being a particular age and following a particular strategy. In memory development, for example, 5-year-olds have been described as not rehearsing, 8-year-olds as rehearsing by repeating single items, and 11-year-olds as rehearsing by reciting the entire list (Naus & Ornstein, 1983). In language development, children at Stage 1 have been said to generate past-tense verb forms by rote, children at Stage 2 by overregularizing the "ed" rule, and children at Stage 3 by knowing both the rule and its exceptions (Brown, Cazden, & Bellugi, 1969). On many Piagetian problems, 5-year-olds have been said to center on single, perceptually salient dimensions, whereas 8-year-olds have been said to coordinate multiple dimensions (Case, 1985).

All these characterizations of the relation between ages and strategies are probably too simple. In each case cited and many others, individual children appear to use multiple strategies rather than a single one (Flavell, Beach, & Chinsky, 1966; Maratsos, 1983; Winer, 1974). These findings suggest that developmental changes are often changes in the distribution of strategy use rather than across-the-board substitutions of one strategy for another. Describing the diverse strategies that children use and the processes that lead to changes in the distribution of strategy use over time promises to make models of cognitive development both more flexible and more realistic.

Relations between initial and end states of development. Another issue raised by the model concerns when patterns of performance that emerge early in development will be maintained and when they will change. The present investigation is not the only one to note parallels in relative problem difficulty between children and adults. The error patterns of children just acquiring a capability have been found to parallel the solution time patterns of adults of much greater skill on tasks as diverse as formation of basic and superordinate categories, analogy problems, word identification, and magnitude comparison (Baron & Strawson, 1976; Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976; Siegler & Robinson, 1982; Sternberg & Rifkin, 1979). An interesting characteristic of these patterns is that they are more easily understood in the error patterns of children than in the solution time patterns of adults. For example, it is easy to understand why first graders should more often err on words with unusual sound-symbol correspondences, such as gone and sword. Except by considering the path of development, it is less easy to understand why adults should take longer to read these words than others of similar length and frequency.

The present model suggests that such parallels between early and later patterns of problem difficulty within a domain occur when backup strategies, connections to related opera-
tions, and presentation frequencies do not change in ways that would alter the pattern of problem difficulty. This framework suggests explanations both for the stability over age of such findings as bone being easier to read than gone and of changes in the relative difficulty of balance scale problems with changes in children's strategies (Baron & Strawson, 1976; Siegler, 1981).

A further implication of the model is that parallels between early and later performance stem from some of the cognitive system's mechanisms for producing performance also producing learning. A recurring experience of developmental psychologists has been that it is much easier to characterize children's thinking of specific points in time than to characterize change processes that operate over time. Part of the difficulty in accounting for transitions may be attributable to the common assumption that transitions arise from different processes than current behavior. The approach taken in this investigation reflects a different assumption. At least in some cases, processes that produce performance may also shape the course of acquisition. Within the present model, children use retrieval and backup strategies to produce performance. The answers that they generate reshape the distributions of associations. The reshaped distributions of associations, in turn, lead to changes in the strategies that children use, the accuracy of their performance, and the speed of their performance. Backup strategies contribute to the transition process in ways that lead to their own demise; retrieval contributes in ways that lead to its own increased use. The lesson may be that exploring relations between current activities and changes over time may enable us to better understand how cognitive changes occur.

References


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