ORIGINS OF COGNITIVE SKILLS

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Out of all of the problem-solving strategies that people could use, how do they decide which ones to use? Even a task as mundane as spelling reveals the variety of strategies that people can employ. Suppose that someone was trying to spell the word “accommodation.” One approach would be to retrieve the spelling. An alternative would be to try to form an image of what the word looks like. Another possibility would be to write out several alternative spellings and to try to recognize the correct one. Yet another strategy would be to look up the word in a dictionary. These variations in strategies are not only “between-subjects” phenomena. Individuals often use each of the approaches at different times, even on a single word.

Good reasons exist for people to know and to use multiple strategies for achieving a goal. Strategies differ in their accuracy, in the amounts of time they require, in their memory demands, and in the range of problems to which they apply. Strategy choices involve tradeoffs among these properties so that people can cope with cognitive and situational constraints. These cognitive and situational constraints can vary from moment to moment, even within what ordinarily is viewed as a single problem. The broader the range of strategies that people know, the more precisely they can shape their approaches to meet these changing circumstances. As becomes evident in the course of this chapter, even young children can choose strategies in adaptive ways. Our goals are to specify how they make such strategy choices, how the ability to make them develops, and what functions the strategy-choice process serves.

The particular strategy choices that we examine are those that preschoolers make in solving simple arithmetic problems. Previous reports indicate that children use a variety of strategies to add and subtract. They count from 1, count on
from the first or the higher number, put up their fingers and count them, tap their feet rhythmically, and decompose complex problems into simpler ones (e.g., 3 + 4 = (3 + 3) = 1). The reports of these visible and audible strategies have been largely anecdotal, but they have been sufficiently persistent to leave little doubt of their existence. Left undescribed, however, have been the mechanisms by which children arrive at a strategy on a particular problem and whether their use of overt strategies results in more effective problem solving. These issues are the focus of the present research.

We begin the chapter by discussing current views on how people select strategies and on how they solve simple addition problems. Then we examine several recent experiments indicating that a single child may use as many as four distinct strategies in solving such problems. The particular strategy that children use on each problem turns out to be closely related to the problem's difficulty. Next we present a distribution of associations model that accounts for this close relation between strategy use and problem difficulty as well as for the existence of the four strategies and for their temporal characteristics. Following this, we present evidence that the model applies to performance on a broader range of addition problems and also to performance on subtraction problems. Then we present a second distribution of associations model that subsumes the first model's assumptions about performance but that also accounts for how strategy choices on simple addition and subtraction problems develop. This second model is expressed as a running computer simulation that initially generates poor performance but that learns from its experience in such a way that it eventually produces patterns of strategy use, solution times, and correct answers and errors much like those of 4- and 5-year-olds. Finally, we speculate about how the revised model might apply to other domains, such as spelling and beginning reading, and discuss its general implications for how people arrive at strategies.

A BRIEF REVIEW OF THE LITERATURES ON STRATEGY CHOICE AND ELEMENTARY ADDITION

The Issue of Strategy Choice

Cognitive psychologists who study adults have devoted considerable effort to determining the strategy people use to perform particular tasks. Out of these efforts have come models of sentence verification (Clark & Chase, 1972; Just & Carpenter, 1975; Trabasso, Rollins, & Shaughnessy, 1971), spatial information processing (Cooper & Shepard, 1973; Shepard & Metzler, 1971), transitive inference (H. H. Clark, 1969; Huttenlocher & Higgins, 1971; Sternberg, 1977), and many other tasks. Recently, however, a number of investigators have noted that people's strategies vary. They have suggested that cognitive theories should
explain how people choose among alternative approaches. Consider three recent comments:

Discussion of the appropriate models for psycholinguistic tasks is usually couched in general terms (i.e., "What models apply to people?"). Our results can be seen as a reminder that this approach is too simplistic. The same ostensibly linguistic task can be approached in radically different ways by different people (MacLeod, Hunt, & Mathews, 1978, p. 506).

Too often psychologists set out to study the way that a task is performed, and miss one of the most interesting and general aspects of human cognitive performance: that there is more than one way to skin a cat. Once we accept this flexibility as a significant characteristic of the way that humans think and learn, rather than a troublesome source of variation in our data, it becomes important to understand the factors that control the adoption of one strategy over others (Farah & Kosslyn, 1982, p. 164).

Information processing psychologists have little to say on "how it is that the child knows what to do" and "what" inside the child's head makes the decisions [Gardner, 1982, p. 421].

These opinions and recommendations have been accompanied by a growing amount of research demonstrating that people do use diverse strategies on tasks for which cognitive psychologists previously had proposed a single model. Hunt and his colleagues demonstrated that different people use different approaches to verify sentences (MacLeod et al., 1978; Mathews, Hunt, & MacLeod, 1980). Cooper and her colleagues described alternative strategies that people use to perform a spatial comparison task (Cooper & Regan, 1982; Glushko & Cooper, 1978). Egan and Grimes-Farrow (1982) and Sternberg and Weil (1980) identified several strategies that people use to draw transitive inferences.

All of these studies of strategy differences followed the comparative approach. That is, they defined groups on some preexisting status variable and demonstrated that group membership predicted behavioral differences. Hunt's, Egan's, Cooper's, and Sternberg's studies all used spatial and verbal abilities as the covariate. People relatively high in spatial ability used one strategy; people relatively high in verbal ability used another. Cross-cultural investigators (e.g., Wagner, 1978) and those interested in aging (e.g., Reder, 1982) have also contrasted the strategies used by members of different groups.

Developmental psychologists have a longer tradition of attending to strategy differences than do psychologists who study adults. One of the central phenomena of developmental psychology is that people of different ages often vary in their approaches. Five-year-olds rarely rehearse when asked to remember
arbitrary lists of words; 8-year-olds usually do rehearse; 11-year-olds rehearse in a more comprehensive, less repetitive, way than 8-year-olds (Flavell, Beach, & Chinsky, 1966; Naus, Ornstein, & Aivano, 1977). Five-year-olds judge which side of a balance scale will go down solely on the basis of weight; 9-year-olds usually consider both weight and distance from the fulcrum but do not know the proportionality rule for combining them; at least some 18-year-olds compute relative torques on the two sides of the balance (Inhelder & Piaget, 1958; Siegler, 1976). Similar strategy differences among different-aged children have been found on analogical reasoning, probability learning, visual scanning, and many other tasks (Sternberg & Rifkin, 1979; Vurpillot, 1968; Weir, 1964).

Investigations showing that people of a particular age, ability profile, or culture tend to use a particular strategy are useful in at least three ways. They document the range of strategies that people can use to solve the problem. They illustrate that different people spontaneously choose different strategies. They show that these choices frequently correlate with group membership. However, they may also divert attention from an issue of even greater interest: how strategy choices are made. A person high in spatial ability need not always use a spatially oriented strategy, nor one high in verbal ability a verbally oriented one. Egan and Grimes-Farrow (1982), Mathews et al. (1980), and Sternberg and Ketron (1982) have all demonstrated that people who ordinarily use one strategy will use a different one if instructed to do so. Cooper and Regan (1982) showed that aspects of the particular stimulus configuration also influence which strategy people adopt. Even if people were not so flexible, they would still need strategy-choice procedures. Before entering the experimental situation, most people do not have extensive experience with the tasks that they encounter (e.g., solving three-term syllogism problems). How does a person high in spatial aptitude know to use a spatially rather than a verbally oriented strategy?

One set of efforts to explain strategy choices is included under the heading of metacognition. Underlying much metacognitive research is the plausible belief that people use explicit knowledge of their cognitive capacities, available strategies, and task demands to determine which strategy to use. When confronted with a problem, they might reason, “This is a difficult problem, too difficult to solve without a powerful strategy such as x, I’d better use x.”

Several difficulties have arisen in metacognitive research that make this mode of explanation less promising than it once appeared. On an empirical level, research has revealed only modest correlations between metacognitive knowledge and performance measures (see reviews by Flavell & Wellman, 1977, and by Cavanaugh & Perlmutter, 1982). On a theoretical level, there is considerable lack of clarity about how metacognitive knowledge would lead to strategy choices. Do people make explicit judgments about their intellectual capacities, about available strategies, and about task demands every time they face a task that they could perform in two or more ways? If not, how do they decide on which tasks to do so? Do they consider every strategy that they conceivably
could use on the task or only some subset of them? If only a subset, how do they decide which ones? How do people know what their cognitive capacity will be or what strategies they could apply when they are presented a novel task? Determining how metacognitive knowledge leads to strategy choices is much more complex than initially might be supposed.

There also appears to be a mismatch between the seemingly useful strategies that very young children at times adopt and their apparent lack of metacognitive knowledge. DeLoache (this volume) reported that when 1½ year olds in a laboratory situation saw an experimenter hide objects that they later needed to find, they engaged in more labeling and other types of discussion of the objects than when the experimenter hid them in the children’s own homes. The children did not discuss the objects at all when they remained visible throughout the waiting period. DeLoache concluded that the children used the labeling and discussion strategy when they needed it to keep alive a memory trace that otherwise might fade. This conclusion seemed consistent with the data. But how did the 1½-year-olds make this decision? Did they possess sufficient knowledge of their memory capacities and of task demands to anticipate that their memory traces might fade if they did not talk about the hidden objects?

In sum, a decade of research on people’s explicit metacognitive knowledge has not explained how they arrive at their strategies. This lack of success raises an intriguing possibility. Perhaps people can arrive at adaptive strategies without explicitly considering capacity limitations, available strategies, and task demands. This possibility will be the focus of the present chapter.

The development of addition skills. Research on how addition skills develop, like research on strategy choice, has followed the comparative approach of equating the performance of one group with one strategy and the performance of another group with another. The two best-known models are those of Groen and Parkman (1972) and Ashcraft (1982). Groen and Parkman proposed the min model. This model indicates that when people are given a problem with two addends, they add by selecting the larger addend and counting up from it the number of times indicated by the smaller. Groen and Parkman hypothesized that the time needed to identify the larger number was a constant for all problems. Therefore, solution times depended only on the number of increments indicated by the smaller number (hence the name min model). The only exception to this formula involved ties, problems with equal augend and addend. Groen and Parkman suggested that answers to these problems were retrieved directly, at a uniformly rapid rate, so that solution times for all tie problems would be faster than solution times for any other problem (excepting those with zero as an addend, which would be retrieved as quickly as ties, due to their not requiring any increments of the larger number).

Several types of evidence supported this model. Groen and Parkman found that the solution times of both first graders and adults increased by constant
amounts with each increase in the minimum number. Svenson (1975) and Svenson and Broquist (1975) found that the model fit the performance of third graders, and, with a small modification, slow-learning older children. Ginsburg (1977) found that children’s verbalizations about their ongoing solution processes often alluded to counting on from the larger number.

Even though Groen and Parkman’s adult data fit the min model, they were reluctant to conclude that adults used the approach. One reason was implausibility; it seemed unlikely that adults had failed to memorize the addition facts after years of using them. A second reason was the shallow slope (20 msec/increment) of the regression equation that fit the adults’ performances. For the min model to apply, the incrementing process would need to be faster than any known elementary information process. Therefore, Groen and Parkman postulated a different model of adult performance. On 95% of trials, adults would retrieve the answer. On the remaining 5%, they would fail to retrieve and instead use the min process. By assuming equally rapid retrieval on all retrieval trials and incrementing times similar to those of children on min process trials, Groen and Parkman were able to fit this model to adults’ performances.

Ashcraft (1982) formulated a different account, which he labeled the fact-retrieval model. Ashcraft’s alternative was motivated by his observation that the magnitude of the squared sum was a better predictor of adults’ solution times than was the magnitude of the minimum number. (Groen and his colleagues do not appear to have examined this predictor variable). To account for the predictive value of sum squared, Ashcraft hypothesized that adults represent addition facts in a form much like a standard addition table, with augends (first numbers) heading each column and addends (second numbers) heading each row. In this mental table, distances between columns and between rows would increase exponentially with increases in the absolute magnitude of the augend and addend. For example, the distance between the third and fourth rows would be greater than that between the second and third rows. Adults would locate the answer to each problem by traveling from the origin to the appropriate augend, traveling down to the appropriate addend, then reading out the sum. Solution times would be directly proportional to distance traveled. The exponential spacing of the rows and columns would lead to a more than linear increase of solution time with increases in the sum. Ashcraft (1982) reported that this model did not fit first graders’ performance as well as did the min model, that the two models fit the performance of third graders equally well, and that the fact-retrieval model fit better the performance of fourth through sixth graders and adults. He therefore concluded that young children use the min approach and that older children and adults use the fact-retrieval model.

Like the previously cited comparative research, these studies of how people solve simple addition problems highlight the fact that people with different demographic characteristics often use different strategies. Also like the previous
research, however, they do not indicate how people arrive at which strategy to use. Further, both models have a somewhat ad hoc flavor. Why should children treat all ties in one way and all nonties in another? Why are columns and rows in the fact-retrieval matrix spaced exponentially? How would children make the transition from the min process to the exponentially spaced matrix?

The models may also underestimate the diversity of children's addition strategies. As noted previously, observational studies have revealed that children sometimes put up their fingers and count them; other times they tap their feet rhythmically; yet other times they count aloud with no obvious referent (Hebbeler, 1976; Ilg & Ames, 1951; Yoshimura, 1974). What is the purpose of using these strategies? What is their relation to the min and fact-retrieval models? How do children decide when to use each approach?

Some Methodological Issues

It is not coincidental that researchers working with chronometric data have tended to postulate a single strategy for all subjects within a given age or ability group, whereas researchers relying on direct observation have postulated multiple strategies. Solution-time data are sufficiently noisy and sufficiently remote from the processes that produce the behavior that strategies can be inferred from them only by aggregating over a large number of trials. Even given a large number of trials, it is extraordinarily difficult to infer from a person's solution times that he or she used multiple strategies. In contrast, simply observing a child can reveal multiple strategies if visible or audible behavior accompanies the strategies. Such observations, however, do not yield sufficiently precise data to allow use of powerful methods for inferring the processes by which children execute each particular strategy.

Videotaping young children's performance seemed to combine the advantages of the two approaches. It would chronicle the behavior occurring between presentation of the problem and statement of the answer, thus allowing assessment of the range of strategies that children used. It would also allow precise measurement of solution times and identification of particular errors. These objective indices would help in inferring how each strategy was executed and how children chose among strategies. Together, the visual record of the strategies and the solution time and error data promised to make the strategy-choice issue tractable.

The possibility that even a single person performing a single task may use a variety of strategies at different times raises several issues. What strategies are used? What are their accuracy and temporal characteristics? How does a person arrive at the strategy to use on a given occasion? Does use of diverse strategies aid the efficiency or accuracy of solutions? If so, how? In the next section, we describe a recent empirical study that addressed these issues.
AN EMPIRICAL STUDY OF CHILDREN'S ADDITION

Method

Siegler and Robinson (1982) examined 4- and 5-year-olds' addition strategies. The 30 children who participated, 17 boys and 13 girls, were students at a university-run preschool.

Children were videotaped as they solved 25 addition problems. The problems were the possible combinations of augends from one to five and addends from one to five. The instructions were the following:

I want you to imagine that you have a pile of oranges. I'll give you more oranges to add to your pile; then you need to tell me how many oranges you have altogether. Okay? You have \( m \) oranges, and I'm going to give you \( n \) to add to your pile. How many do you have altogether?

After four or five questions, many children indicated that they preferred to hear the problem in the form "How much is \( m + n \)?" We complied with their request.

Each child was presented each problem on two occasions; thus, children were eventually presented 50 trials. These 50 trials were divided among six sessions, with eight or nine problems in each session. Children performed the problems while sitting at a desk with a bare top; no external objects were present for them to manipulate while they solved the problems. The experimenter praised the children and gave each child a star following each correct answer. Sessions lasted approximately 5 minutes apiece.

Results

The videotapes revealed four strategies. Three were overt (visible or audible) approaches. Sometimes children raised fingers corresponding to each addend and counted them (the counting-fingers strategy). Other times they lifted fingers corresponding to each addend but answered without counting them (the fingers strategy). Yet other times they counted aloud (or moved their lips in a visible, silent-counting sequence), but their counting did not have any obvious external referent (the counting strategy). The fourth approach involved no visible or audible behavior. For reasons that become apparent in the course of the chapter, we labeled this the retrieval strategy.

As shown in Table 9.1, the four strategies differed in their frequency of use and in their temporal and accuracy characteristics. Of particular interest within the analysis that follows were relative solution times. For each pair of strategies, we compared mean solution times on each of the 25 problems for those trials on which children used one strategy to the mean solution times on trials on that problem on which they used the other. (Only the solution times of children who
TABLE 9.1
Characteristics of Arithmetic Strategies (Siegl & Robinson, 1982)

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Trials on Which Strategy Used (%)</th>
<th>Mean Solution Time (Sec)</th>
<th>Correct Answers (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting fingers</td>
<td>15</td>
<td>14.0</td>
<td>87</td>
</tr>
<tr>
<td>Fingers</td>
<td>13</td>
<td>6.6</td>
<td>89</td>
</tr>
<tr>
<td>Counting</td>
<td>8</td>
<td>9.0</td>
<td>54</td>
</tr>
<tr>
<td>Retrieval</td>
<td>64</td>
<td>4.0</td>
<td>66</td>
</tr>
</tbody>
</table>

used both strategies being compared at least twice were included in this analysis). Retrieval was significantly faster than the fingers strategy, *t*(24) = 3.10, which in turn was faster than the counting-fingers strategy, *t*(23) = 3.96. Retrieval was also significantly faster than counting, *t*(23) = 8.87 which in turn was faster than counting fingers, *t*(23) = 3.91 (all *p*'s < .01).1

The most intriguing finding of the experiment was unexpected. The preschoolers proved to be surprisingly adept at matching their use of strategies with the difficulty of the problems. There was a very close association (*r* = .91) between percentage of errors on the 25 problems and percentage of use of the three overt strategies on them (Fig. 9.1). Children most frequently used overt strategies on exactly those problems that were the most difficult to answer correctly. Percentage of overt strategy use on each problem was a better predictor of percentage of errors on that problem than were any of the other variables that we included in the regression analysis: the sum, the larger number, the smaller number, the first number, the second number, the square of the sum, or the min model. (For details of this and subsequent regression analyses, see Appendix A.)

The relation between overt strategy use and errors was not a simple causal one in which use of overt strategies caused children to err. Viewing each problem individually, on 24 of the 25 problems children erred on a lower percentage of trials on which they used overt strategies than on trials on which they did not, *t*(24) = 6.87, *p* < .01. For example, on the problem 3 + 4, children erred on 31% of trials on which they used overt strategies versus 80% of trials on which they did not.

Children's use of overt strategies was also closely related to a second measure of problem difficulty, mean solution times. The longer the mean solution time on a problem, the higher the percentage of overt strategy use on that problem (*r* = .90). The relation could not be explained solely as the overt strategies' taking longer to execute (although they did). Even when we excluded from our calculation of mean solution times those trials on which children used overt strategies, the relation remained substantial. Specifically, the correlation between mean

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1No children used the counting or the counting-fingers strategy on the problem 1 + 1, thus leading to 23 rather than 24 degrees of freedom on comparisons involving those strategies.
solution times on retrieval trials on each problem and percentage of overt strategy use on that problem was $r = .76$.

To summarize, children used four strategies: fingers, counting fingers, counting, and retrieval. Retrieval was the fastest strategy, fingers the next fastest, and counting and counting fingers the slowest. Use of the overt strategies helped

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$^{2}$In computing the mean solution times on all trials, the two longest and two shortest times were ignored. These four times were also excluded in the computation of the mean solution time for each strategy (i.e., if two of the four longest trials were retrieval trials, these two would be excluded from the computation of the mean solution time on retrieval trials). The reason for doing this was that the attention of the 4- and 5-year-olds sometimes drifted, leading to long pauses or to extremely rapid answers that seemed intended "to get it over with."
children solve problems. On 24 of the 25 problems, children added more accurately on trials on which they used overt strategies than on trials on which they did not. Finally, strong relations among the percentage of errors, the mean solution time, and the percentage of overt strategy use on each problem indicated that children had some systematic way of choosing when to use overt strategies. We next consider how children might have arrived at their strategies.

A DISTRIBUTION OF ASSOCIATIONS MODEL OF STRATEGY CHOICE

Fig. 9.2 outlines a model of how children generated their addition performance. We have labeled it the distribution of associations model, because within it errors, solution times, and overt strategy use are all functions of a single variable: the distribution of associations between problems and potential answers.

The model includes a representation and a process. The representation consists of associations of varying strengths between each problem and possible answers to the problem. The numerical values in the Fig. 9.2A matrix are the estimated strengths of these associations. For example, an associative strength of .05 links the problem 1 + 1 and the answer "1," and an associative strength of .86 links 1 + 1 and "2."

The process that operates on this representation can be divided into three phases: retrieval, elaboration of the representation, and counting. As shown in Fig. 9.2B, the child (who we here imagine as a boy) first retrieves an answer. If he is sufficiently confident of it, he states it. Otherwise, he next generates a more elaborate representation of the problem, perhaps by putting up fingers, and tries again to retrieve an answer. As before, if he is sufficiently confident of the answer he states it. Otherwise, he counts the objects in the representation and states the last number as the answer.

Now we can examine the process in greater detail. The first phase (Steps 1 to 8) involves an effort at retrieval. The child sets two parameters: a confidence criterion and a search length. The confidence criterion defines a value that must be exceeded by the associative strength of a retrieved answer for the child to state that answer. The search length indicates the maximum number of retrieval efforts the child will make before moving on to the second phase of the process. Once

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3These estimated strengths were derived from performance in a separate "overt-strategies-prohibited" experiment. In this experiment, 4-year-olds were presented the Siegler and Robinson procedure except that they were explicitly asked to "just say what you think the right answer is without putting up your fingers or counting." The purpose of these instructions was to obtain the purest possible estimate of the strengths of associations between problems and answers. Each associative strength in the Fig. 9.2A matrix corresponds to the proportion of trials on which children advanced the particular answer to the particular problem in the overt-strategies-prohibited experiment.
these parameters are set, the child retrieves an answer. The probability of any
given answer’s being retrieved on a particular retrieval effort is proportional to
the associative strength of that answer for that problem. Thus, the probability of
retrieving “2” as the answer to “1 + 1” would be .86 (Fig. 9.2A). If the
 associative strength of the retrieved answer exceeds the confidence criterion, the
child states that answer. Otherwise, the child examines whether the number of

A. Representation (Associative Strengths)

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FIG. 9.2. The strategy choice model. In Fig. 9.2B, “answer,” refers to whichever answer is retrieved on the particular retrieval effort. Also in Fig. 9.2B “problem-answer, associative strength” refers to the association between the elaborated representation and the retrieved answer.
searches that have been conducted is within the permissible search length. If so, the child again retrieves an answer, compares it to the confidence criterion, and advances it as the solution if its associative strength exceeds the criterion. Retrieval efforts continue as long as the associative strength of each retrieved answer is below the confidence criterion and the number of searches does not exceed the search length. If the point is reached at which the number of searches does exceed the search length, the child proceeds to the second phase.
In the second phase, the child creates an elaborated representation of the problem. This can be either an elaborated external representation, for example one in which the child puts up his fingers, or an elaborated internal representation, for example one in which the child forms a mental image of objects corresponding to augend and addend. Putting up fingers or forming an image adds visual associations between the elaborated representation and various answers to the already-existing association between the problem and various answers. If the elaborated representation involves the child’s fingers, it adds kinesthetic associations as well. We refer to these visual and kinesthetic associations as elaborated representation-answer associations as opposed to the problem-answer associations discussed previously. Having formed the elaborated representation, the child again retrieves an answer. If that answer’s associative strength exceeds the confidence criterion, the child responds. If it does not, the child proceeds to the third phase, an algorithmic process in which he or she counts the objects in the elaborated representation and advances the number assigned to the last object as the sum.

It may be useful to examine how a child using the model would solve a particular problem. Suppose a girl was presented the problem “3 + 4.” Initially, she chooses a confidence criterion and a search length. For purpose of illustration, we assume that she selects the confidence criterion .50 and the search length two. Next, she retrieves an answer. As shown in Fig. 9.2A, the probability of retrieving 3 is .05, the probability of retrieving 4 is .11, the probability of retrieving 5 is .21, and so on. Suppose that the child retrieves 5. This answer’s associative strength, .21, does not exceed the current confidence criterion, .50. Therefore, the girl does not state it as the answer. She next checks whether the number of searches has reached the search length. Because it has not, she again retrieves an answer. This time she might retrieve 7. The associative strength of 7, .29, does not exceed the confidence criterion, .50. Because the number of searches, two, has reached the allowed search length, the child proceeds to the second phase of the process.

In this second phase, the girl initially represents the problem either by forming a mental image or by putting up fingers. We assume that she puts up three fingers on one hand and four on the other. Next, she again retrieves an answer. As indicated in Footnote 4, combining the problem-answer and the representation-answer associative strengths increases the child’s probability of retrieving 7

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4The probability of a given answer’s being retrieved at this point is determined by adding the problem-answer and the elaborated representation-answer associative strengths and dividing by one plus the elaborated representation-answer associative strength. In our computer simulation (described later), we arbitrarily decided that each external representation added a constant .05 to the answer corresponding to the number of objects in the representation. Thus, if the problem 1 + 4 was represented with five fingers, and the initial associative strength of 1 + 4 = 5 was .61, the new associative strength would be .66/1.05 = .63.
from .29 to .32. Suppose that she retrieves 7. Its associative strength still does not exceed the .50 confidence criterion. Therefore, the child does not state it. She instead proceeds to the third phase of the process. Here, she counts her fingers and states the last number as the answer to the problem. If she counts correctly, she will say "7."

This model accounts for the strategies that children use, for the temporal characteristics of the strategies, and for the close relations among the percentage of overt strategy use, the percentage of errors, and the mean solution times on each problem. First consider how it accounts for the existence of the four strategies. The retrieval strategy appears if children retrieve an answer whose problem-answer associative strength exceeds their confidence criterion (Steps 1 to 5, sometimes Steps 6 and 7, Step 8). The fingers strategy emerges when children fail to retrieve an answer whose problem-answer associative strength exceeds their confidence criterion, put up their fingers, and then retrieve an answer in which the sum of the problem-answer and the elaborated representation-answer associative strengths exceeds their confidence criterion (Steps 1 to 7, 9 to 10, 12 to 14). The counting-fingers strategy appears if children fail to retrieve an answer whose problem-answer associative strength exceeds their confidence criterion, put up their fingers, fail to retrieve an answer in which the sum of the elaborated representation-answer and problem-answer associative strengths exceeds the confidence criterion, and finally count their fingers (Steps 1 to 7, 9 to 10, 12 to 13, 15 to 17). The counting strategy is observed if children fail to retrieve an answer whose problem-answer associative strength exceeds their confidence criterion, form an elaborated internal representation, fail to retrieve an answer in which the sum of the elaborated representation-answer and problem-answer associative strengths exceeds the confidence criterion, and finally count the objects in the internal representation (Steps 1 to 7, 9, 11 to 13, 15 to 17).

The relative solution times of the strategies arise because the faster strategies are component parts of the slower ones. To use the fingers strategy, children must execute all of the steps in the retrieval strategy and four additional ones. To execute the counting-fingers strategy, children must proceed through all of the steps in the fingers strategy and two additional ones. To execute the counting strategy, children must execute all of the steps in the retrieval strategy plus six others. If we can equate the time needed to form elaborated internal and elaborated external representations, children using the counting strategy must execute all of the steps in the fingers strategy plus two others. Thus, the retrieval strategy should be faster than any of the other strategies, the fingers strategy should be

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5Any trials on which children generated an internal representation and then stated the answer that they retrieved (Steps 1 to 7 and 10 to 13) would also be classified as retrieval trials. This path was expected to be rare, however. Kinesthetic cues would not be available to mediate the elaborated representation-answer association, and visual cues would be weaker than if the objects in the elaborated representation were visible.
faster than the counting-fingers strategy, and, if the time needed to form an external representation does not exceed the time needed to form an internal one, the fingers strategy also should be faster than the counting strategy.⁶

Perhaps the most important feature of the model is that it generates close associations among percentage of errors, mean solution time, and percentage of overt strategy use on each problem. The associations arise because all three dependent variables are functions of the same independent variable: the distribution of associations linking problems and answers. The way in which this dependency operates becomes apparent when we compare the outcomes of a peaked distribution of associations, such as that for 2 + 1 in Fig. 9.3, with those of a flat distribution, such as that for 3 + 4. A low percentage of use of overt strategies, a low percentage of errors, and a short mean solution time all accompany the peaked distribution. Relative to the flat distribution, the peaked distribution results in: (1) less frequent use of overt strategies (because the answer that is retrieved is more likely to have high associative strength, which allows it to exceed more confidence criteria, thus leading to use of retrieval rather than overt strategies); (2) fewer errors (because of the higher probability of retrieving and

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⁶The relative accuracy of the four strategies can also be explained within the framework of the model, though the explanation involves several considerations external to the model. See Siegler and Robinson (1982, pp. 298–299) for this explanation.
stating the answer that forms the peak of the distribution, which is generally the correct answer; and (3) shorter solution times (because the probability of retrieving on an early search an answer whose associative strength exceeds any given confidence criterion is greater the more peaked the distribution of associations).

At least two nonintuitive predictions follow from this model. One is that the correlation between percentage of errors on each problem and percentage of overt strategy use on that problem is primarily a correlation between the percentage of errors on retrieval trials on each problem and the percentage of overt strategy use on the problem. That is, the correlation does not depend on the percentage of errors on counting, counting-fingers, and fingers strategy trials. The reasoning underlying this prediction is that percentage of overt strategy use on a problem and percentage of errors on retrieval trials on that problem both depend entirely on the distribution of associations, but percentage of errors on nonretrieval trials on the problem depends on other factors.

This logic may become clearer when we examine the model's account of the way in which errors are produced by each of the four strategies. On retrieval trials, the percentage of errors on each problem depends only on the distribution of associations. Errors are made when an incorrect answer retrieved from the distribution exceeds the confidence criterion. The flatter the distribution, the greater the proportion of retrieval trials on which this will happen. The distribution's flatness increases both the likelihood of an incorrect answer's being retrieved and the likelihood of its having sufficient associative strength to be stated. Thus percentage of errors on retrieval trials, like percentage of overt strategy use, depends entirely on the distribution of associations.

In contrast, the percentage of errors on counting and counting-fingers trials is unaffected by the distribution of associations. The counting and counting-fingers strategies arise when children fail to retrieve a statable answer from the distribution of associations. Instead, they base answers on their counts of the objects in their elaborated representations. Errors are made when they misrepresent the number of objects in the problem or when they miscount them. The greater the sum, the more objects that can be misrepresented or miscounted, and therefore the greater the likelihood of errors. Thus, colinearity with the sum should account for whatever correlation emerges between the percentage of errors on counting and counting-fingers trials on each problem and the percentage of overt strategy use on the problem.

Correlations involving the percentage of errors on fingers-strategy trials should occupy a middle ground. Recall that the fingers strategy is produced when children first fail to retrieve an answer whose associative strength exceeds their confidence criterion, then elaborate the representation by putting up their fingers, and then retrieve an answer in which the sum of the problem-answer and the elaborated representation-answer associative strengths exceeds the confidence criterion. Under such circumstances, the percentage of errors is a function of both the distribution of associations between problems and answers and the
distribution of associations between elaborated representations and answers. A relatively peaked distribution of associations increases the probability that the problem-answer and elaborated representation-answer associations together will lead to retrieval of an answer whose associative strength exceeds the confidence criterion. However, the sum also may influence this likelihood. Presumably, children more often correctly represent the addends on problems with small sums, which leads to the elaborated representation-answer association’s more often being added to the correct rather than to an incorrect answer on these problems. Thus, the percentage of errors on fingers-strategy trials on each problem should correlate somewhat with percentage of overt strategy use on that problem, but not as highly as percentage of errors on retrieval trials on the problem.

The logic of these predictions can be summarized as follows:

A. If percentage of overt strategy use on each problem is a function of only the distribution of associations; and
B. If percentage of errors on retrieval trials on each problem is also a function of only the distribution of associations; and
C. If percentage of errors on fingers trials is in part a function of the distribution of associations; and
D. If percentage of errors on counting and counting-fingers trials is not at all a function of the distribution of associations, instead being a function of the sum;

Then

1. Percentage of errors on retrieval trials should correlate highly with percentage of overt strategy use.
2. Percentage of errors on counting and counting-fingers trials should correlate less highly with percentage of overt strategy use, especially when the contribution of the sum is partialed out.
3. Percentage of errors on fingers trials should show an intermediate degree of correlation with percentage of overt strategy use.

The data were entirely consistent with these predictions. As shown in Fig. 9.4, percentage of errors on retrieval trials on each problem correlated \( r = .92 \) with percentage of overt strategy use on the problem. This correlation was actually slightly higher than the correlation reported earlier between the percentage of errors on all trials and the percentage of overt strategy use. The correlation between percentage of errors on counting and counting-fingers trials on each problem and the percentage of overt strategy use on that problem \( (r = .38) \) was significantly lower, \( t(22) = 6.47, p < .01 \). Yet more striking was the difference between the partial correlations. When the contribution of the sum was partialed out, percentage of overt strategy use correlated \( r = .87 \) with percentage of errors on retrieval trials. The corresponding partial correlation between percentage of
FIG. 9.4. Correlations between percentage of overt strategy use on each problem and percentage of errors on all trials, retrieval trials, and counting and counting-fingers trials. Data from Siegler and Robinson (1982).
overt strategy use and percentage of errors on counting and counting-fingers trials was \( r = -0.23 \). The difference between the two partial correlations was highly significant \( t(22) = 7.45, p < .01 \).

Also as predicted, the correlation between percentage of errors on fingers trials and percentage of overt strategy use on each problem was in between \( r = 0.68 \). It did not differ significantly from either of the other correlations. When the contribution of the sum was partialed out, the correlation became \( r = 0.66 \). This correlation was significantly lower than the one involving errors on retrieval trials \( t(22) = 2.10, p < 0.05 \) and significantly higher than the one involving errors on counting and counting-fingers trials \( t(22) = 5.08 \).\(^7\) The findings supported the view that the original correlation between errors and overt strategy use was due primarily to overt strategy use and errors on retrieval trials being functions of the same variable, the distribution of associations, and of errors on overt strategy trials depending on other variables.

Similar logic can be applied to analyzing the correlation between solution times and overt strategy use. Solution times on retrieval trials should derive exclusively from the distribution of associations. The more peaked this distribution, the more quickly children should retrieve an answer whose associative strength exceeds their confidence criterion. On counting and counting-fingers trials, the amount of time needed to generate an elaborated representation and to count the objects in it would influence the times. These would depend on the number of objects that need to be represented and counted—in short, on the sum. The model made no direct prediction concerning solution times on fingers trials on each problem. Thus, the prediction of the model was that the correlation between percentage of overt strategy use and mean solution times on retrieval trials on each problem would be higher than the correlation between percentage of overt strategy use and mean solution times on counting and counting-fingers trials. The pattern would be most evident with the contribution of the sum partialed out from both correlations.

A multiple regression analysis indicated that, as predicted by the model, percentage of overt strategy use was the most powerful predictor of mean solution times on retrieval trials on the 25 problems \( r = 0.76 \) (Appendix A). Contrary to expectation, however, the best predictor of solution times on counting and counting-fingers trials on each problem was also percentage of overt strategy use \( r = 0.83 \). The partial correlations showed the same pattern. With the contribution of the sum partialed out, the correlation between percentage of overt strategy use on each problem and mean solution times on retrieval trials on the

\(^7\)This finding argued against the possibility that the difference in the magnitudes of the correlations involving errors on retrieval trials and those involving errors on counting and counting-fingers trials was due to the latter’s being based on fewer trials per problem. If trials per problem were the key variable, we would not expect percentage of overt strategy use to correlate more highly with the percentage of errors on fingers trials than with the percentage of errors on counting and counting-fingers trials. There were fewer fingers trials than counting and counting-fingers trials.
problem was \( r = .68 \). The corresponding partial correlation between overt strategy use and solution times on counting and counting-fingers trials was \( r = .71 \).\(^8\)

In summary, all but one of the model’s predictions were consistent with the data. The model accounted for the four strategies that children used, the relative solution times of the strategies, the correlations among percentage of errors, mean solution times, and percentage of overt strategy use on each problem, and the source of at least the correlation between errors and strategy use’s being percentage of errors on retrieval trials. The model also possessed several other properties that seemed desirable. It allowed children to strike a balance between speed and accuracy demands. When possible, they would use the relatively rapid retrieval approach. When retrieval yielded no answer that was sufficiently strongly associated with the problem, they would fall back on successively more time-consuming overt approaches. The model also had the advantage of treating all problems in the same way. It did not assume that ties have a special status or that the mental distance between sums increases exponentially with their sizes. Finally, as is discussed in more detail later in the chapter, the model suggests how development might occur. As children’s distributions of associations become increasingly peaked, they rely increasingly on retrieval, advance the correct answer more often, and answer more quickly. In short, their performance becomes increasingly adultlike.

A MATHEMATICAL EXPRESSION OF THE MODEL

We wanted to provide a rigorous test of the sufficiency of the model to produce strong relations among overt strategy use, frequency of errors on retrieval trials, and length of mean solution times on each problem. Therefore, we translated the model’s predictions into algebraic equations, inserted a large range of parameter values into the model, and compared the model’s behavior to that that children had displayed.

The following equations were used to describe, for each problem, the probability of retrieving an answer that exceeded the confidence criterion, the probability of overt strategy use, the probability of an error on a retrieval trial, and the expected solution time on a retrieval trial.

Probability of Retrieving Answer on a Problem that Exceeds Confidence Criterion =

\[
R = \sum_{a=1}^{A} (AS_a)(p(AS_a > CC))/(\sum_{a=1}^{A} AS_a)
\]

\(^8\)As becomes evident later, patterns of solution times in all subsequent experiments conformed to the predictions of the model. Therefore, no effort was made to explain why the solution times in this experiment did not.
Probability of Overt Strategy Use on a Problem = \((1 - R)^N\)

Probability of Error on Retrieval Trials on Each Problem =

\[1 - ((p(AS_{ca} > CC))(AS_{ca})/\sum_{a=1}^{A} (AS_a(p(AS_a > CC))))\]

Expected Solution Time on Retrieval Trials on Each Problem =

\[\sum_{n=1}^{N} nR((1 - R)^{n-1}) + N(1 - (\sum_{n=1}^{N} R((1 - R)^{n-1})))\]

where \(R\) refers to the probability that the answer retrieved on any given search will exceed the confidence criterion, \(AS_a\) refers to the associative strength of answer \(a\), \(CC\) refers to the confidence criterion, \(N\) refers to the search length, and \(ASca\) refers to the associative strength of the correct answer.

The correspondence between each equation and the process it models is quite straightforward. The probability on each problem of retrieving an answer that exceeds the confidence criterion is the sum of the associative strengths of answers to that problem that exceed the criterion divided by the sum of the associative strengths of all answers to the problem. The probability of overt strategy use on a problem is the probability that none of the searches will an answer be retrieved that exceeds the confidence criterion. The probability of an error on a retrieval trial is the probability of retrieving an incorrect answer whose associative strength exceeds the confidence criterion divided by the probability of retrieving a correct or incorrect answer whose associative strength exceeds the confidence criterion. The expected solution time on retrieval trials is proportional to the expected value of the number of searches on each problem before an answer is stated.

We examined the operation of this mathematical model under the 72 possible combinations of confidence criteria (.05, .10, .15, .20, .30, .40, .50, .60, .70, .80, .90, 1.00) and search lengths (one to six). For each confidence-criterion–search-length pair, we applied the four equations to each of the 25 problems that children had been presented. Then we combined the results to obtain expected percentages of errors on retrieval trials, mean solution times on retrieval trials, and percentages of overt strategy use.

The model was tested in two ways, corresponding to measures of internal and external validity. First we wanted to establish the sufficiency of the equations to generate the high correlations among the three variables that we had observed. To do this, we entered into the equations associative strengths (operationally defined here as the relative frequencies of answers given on retrieval trials in Siegler and Robinson) and used the output of the equations to estimate percentages of errors, mean solution times, and percentages of overt strategy use on
each problem. If the equations operated as anticipated, this procedure would produce high intercorrelations among the expected values for the three variables. The equations passed this test. The intercorrelations among the output of the equations for errors, solution times, and overt strategy use ranged from \( r = .92 \) to \( r = .99 \).

As another measure of internal validity, we tested whether the model's predictions for each measure would correlate highly with the children's mean solution times, percentage of overt strategy use, and percentage of errors in the Siegler and Robinson experiment. The correlation between modeled and observed percentage of errors on retrieval trials was \( r = .94 \), between modeled and observed frequency of overt strategy use \( r = .89 \), and between modeled and observed mean solution times on retrieval trials \( r = .92 \).

We next tested the mathematical model's ability to predict across data sets. We used the associative strengths displayed in Fig. 9.2A to predict performance in the Siegler and Robinson experiment. As indicated in Footnote 2, these associative strengths were estimated from the performance of different children under somewhat different experimental conditions (no overt strategies allowed) than the data being predicted. The correlation between the predicted and observed percentage of overt strategy use on each problem was \( r = .87 \). The correlation between the predicted and observed percentages of errors on each problem was \( r = .77 \). The correlation between the predicted and observed mean solution times on each problem was \( r = .83 \). These results demonstrated that estimates of associative strengths obtained in one experiment could be used to predict experimental data in another.

A REPLICATION AND EXTENSION EXPERIMENT

Many results from the initial experiment were unanticipated. Also, it seemed possible that the generality of the findings would be limited to problems in which the fingers and counting-fingers strategies were easy to use—that is, problems with addends no greater than 5 and/or sums no greater than 10. We therefore performed a second experiment replicating the initial condition and adding problem sets on which overt strategies would be more difficult to execute.

A further purpose of the experiment was to test an alternative to the Fig. 9.2 model of strategy choice. In this alternative, the close connection between errors, solution times, and overt strategy use arises because children explicitly judge each problem's difficulty and use an overt strategy when they judge the difficulty to be high. Thus:

\[
\text{problem difficulty} \rightarrow \text{judgments of problem difficulty} \rightarrow \text{overt strategy use}
\]

This depiction suggests that judgments of problem difficulty should correlate highly with both actual problem difficulty (as measured by percentage of errors
on each problem) and overt strategy use. Otherwise the observed high correlation between percentage of errors and overt strategies would be difficult to explain.

In a preliminary test of this alternative, Siegler and Robinson (1982) asked a group of 5 year olds, students at a nursery school similar to the one at which the original experiment had been run, to label each of the 25 problems as hard, easy, or in between. "Hard" ratings were quantified as 2, "easy" ratings as 0, and "in-between" ratings as 1. It was found that the mean difficulty ratings on each of the 25 problems correlated $r = .47$ with the percentage of errors on the problem and $r = .51$ with the percentage of overt strategy use on the problems. These correlations were substantially lower than the previously noted $r = .91$ correlation between percentage of overt strategy use on each problem and percentage of errors on the problem. Because the judgment data were collected from different children than the addition performance data, however, the experiment provided only a preliminary index. The replication and extension experiment, in which both types of data were obtained from the same children, would provide a more definitive test.

Method

The 42 children who participated, 23 boys and 19 girls, attended either a university preschool or a nursery school in a middle-class area of Pittsburgh. In each of the three conditions, there were eight 4 year olds and six 5-year-olds.

Children in Group 1 of this experiment (the replication condition) were presented the same 25 problems as the children studied by Siegler and Robinson. These were all of the problems on which both addends were less than or equal to 5 and on which the sum was less than or equal to 10. Children in Group 2 were presented 25 problems on which the sum again was less than or equal to 10 but on which either addend could be as great as 9. Children in Group 3 were presented a third set of 25 problems; in this set, the sum could be as high as 12 and the addends as large as 11. To maintain the interest of children who were not especially skillful at adding, Groups 2 and 3 included two subsets of problems. In each group, 13 of the problems were selected from the relatively easy items presented in Group 1 in which addends never exceeded 5 and sums never exceeded 10. The remaining 12 problems in Group 2 all had one addend between 6 and 9 and a sum that did not exceed 10. The remaining 12 problems in Group 3 all had one addend between 6 and 11 and sums of 11 or 12.

The instructions began in the same way as those used by Siegler and Robinson. However, to ensure that all children knew that they could use overt strategies, we added the following instructions at the end: "You can do anything you want to help you get the right answer. If you want to use your fingers or count aloud, that's fine." It seemed possible that these instructions would increase overall use of the overt strategies, but unlikely that they would influence the correlation between use of overt strategies and problem difficulty.
Within a week of their last counting session, children in all three groups were presented the metacognitive judgment task that peers in the Siegler and Robinson study had performed. They were told, "Remember the problems I asked you about? Well, some of those problems were easy, some were hard, and some were in between easy and hard. I'm going to ask you about each problem and you tell me whether you think that it is easy, hard, or in between."

Results

Existence of Strategies. The model suggested that children would use four strategies: counting fingers, fingers, counting, and retrieval. As shown in Table 9.2, children in all groups used each of these approaches. We had anticipated that the new instructions' explicit statement that children could use overt strategies might result in more frequent use of such strategies. The data showed little tendency in this direction, however. In the Siegler and Robinson study, children used overt strategies on 36% of trials; in the replication and extension experiment, they used them on 43%. These percentages did not differ significantly, \( t < 1 \).

Children in Group 3 encountered items on which the sum exceeded their number of fingers. This situation appeared to lead them to adopt new procedures, because on 8% of trials they used strategies that did not fall neatly into the four categories. The two most common variants, each accounting for 3% of total trials, were counting/counting fingers on and counting fingers/fail. In the first of these, children would count up to one addend without putting up their fingers, then put up their fingers to represent the other addend, and then count their fingers starting with the number one greater than the result of their initial counting procedure. As long as the addend that children represented on their fingers was no greater than 10, this procedure circumvented the difficulty of the sum's exceeding their number of fingers. The second relatively common procedure involved representing as many numbers as the child had fingers, counting all of them, and then arbitrarily naming a larger number if both addends had not been totally represented. Several other procedures were used very occasionally, none of them exceeding 1% of total trials. Like the procedures just described, these

\[9\text{A variant of the counting strategy that was not observed by Siegler and Robinson appeared in this experiment. On some trials, children put up fingers synchronously with counting out the sum. That is, they put up a first finger while saying "1," a second finger while saying "2," and so on. These trials could not be classified as counting-fingers trials, because there was no evidence that the fingers were being used to represent the addends. Unlike the trials that were labeled counting fingers, children did not represent the addends separately from counting out the sum, did not pause between addends, and did not represent addends on separate hands. Accuracy rates on these trials were also more similar to those observed on counting than on counting-fingers trials. Therefore, these approaches were viewed as a form of counting in which fingers were incidental accompaniments, much as tapping one's feet might be.}\]
involved combining the usual strategies in innovative ways, and appeared aimed at overcoming the difficulty associated with sums greater than 10.

Relative Accuracy of Strategies. The relative accuracy of the four strategies was identical to that found in the earlier study. In all three groups, the fingers strategy was the most accurate, the counting-fingers strategy the next most accurate, and the retrieval and counting strategies the least accurate. Also as previously, children performed more accurately on those trials on each problem on which they used overt strategies than on those trials on which they did not (for Group 1, $t(24) = 5.14$; for Group 2, $t(24) = 2.89$; for Group 3, $t(24) = 3.61$; all $p$’s < .01). Thus, using the overt strategies seemed to help children solve the problems.

Relative Solution Times of Strategies. The model predicted that retrieval would be the fastest strategy, that fingers would be the next fastest, and that counting and counting fingers would be the slowest. The data were in accord with these predictions. As shown in Table 9.2, the retrieval and fingers strategies were substantially faster than the counting and counting-fingers strategies in each of the three groups. Including those children from all three groups who used both of the pairs of strategies at least twice, retrieval was significantly faster than the fingers strategy, $t(18) = 2.34$, $p < .05$, which in turn was significantly faster.
than the counting strategy, $t(14) = 2.42, p < .05$, or the counting-fingers strategy, $t(13) = 3.53, p < .01$.\footnote{The reason for comparing the solution times of children in all three groups in a single analysis, rather than comparing the times in each group separately, was to obtain enough children who used each pair of strategies to make a statistical comparison reasonably powerful. In many cases, only five or six children in each group used a particular pair of strategies.}

*Relations among Errors, Solution Times, and Overt Strategy Use.* The model predicted that the percentage of errors, mean solution times, and percentage of overt strategy use on each problem would vary together. The expected pattern emerged in all three groups. In Group 1, percentage of overt strategy use on each problem correlated $r = .79$ with percentage of errors on that problem. In Group 2, the two variables correlated $r = .79$. In Group 3, they correlated $r = .81$. The relation is illustrated in Fig. 9.5 and in Appendix A.

A similar relation was present between overt strategy use and solution times. In Group 1, mean solution times correlated $r = .91$ with percentage of overt strategy use; in Group 2, the two variables correlated $r = .81$; and in Group 3, the two variables correlated $r = .92$. As in Siegler and Robinson, the relations were present even when only solution times on retrieval trials were considered: $r = .80, r = .75$, and $r = .90$ for Groups 1, 2, and 3, respectively (Appendix A).

*The Source of the Relations among Strategy Use, Errors, and Solution Times.* The model predicted that the correlations among these three variables derived primarily from percentage of errors and mean solution times on retrieval trials. This prediction again proved accurate. First consider analyses involving errors on retrieval trials. In each of the three groups, the best predictor of the percentage of errors on retrieval trials on each problem was the percentage of overt strategy use on that problem (Appendix A). The correlations ranged from $r = .80$ to $r = .88$ (Table 9.3). In all three cases, as in the Siegler and Robinson data, these correlations were greater than the correlations involving the overall percentage of errors. When the contribution of the sum was partialled out, the correlation between errors on retrieval trials and overt strategy use remained quite high, ranging from $r = .56$ to $r = .70$. The correlations involving percentage of errors on counting and counting-fingers trials showed a different pattern. In none of the three groups was overt strategy use the best predictor of percentage of errors on these trials. The raw correlations ranged from $r = .37$ to $r = .51$. With the contribution of the sum partialled out, the correlations ranged from $r = .23$ to $r = .42$.

For each of the three groups, the raw correlation between percentage of errors on retrieval trials and percentage of overt strategy use was significantly greater than the corresponding correlation between percentage of errors on counting and counting-fingers trials and percentage of overt strategy use. The difference be-
FIG. 9.5. Correlations between percentage of overt strategy use on each problem and percentage of errors on all trials, retrieval trials, and counting and counting-fingers trials. Data from replication and extension experiment.
### TABLE 9.3
Source of Correlations of Overt Strategy Use with Errors and Solution Times: Replication and Extension Experiment\(^a\)

<table>
<thead>
<tr>
<th></th>
<th>Errors</th>
<th>Counting and</th>
<th>Solution Times</th>
<th>Counting and</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Retrieval Trials</td>
<td>Fingers Trials(^b)</td>
<td>Counting-Fingers Trials</td>
<td>Retrieval Trials</td>
</tr>
<tr>
<td>Group 1</td>
<td>.83</td>
<td>.51</td>
<td>.79</td>
<td>.42</td>
</tr>
<tr>
<td>Group 2</td>
<td>.80</td>
<td>.68</td>
<td>.75</td>
<td>.54</td>
</tr>
<tr>
<td>Group 3</td>
<td>.88</td>
<td>.37</td>
<td>.84</td>
<td>.79</td>
</tr>
</tbody>
</table>

#### B. Partial Correlations (Correlations with the Sum Partialled Out)

<table>
<thead>
<tr>
<th></th>
<th>Errors</th>
<th>Counting and</th>
<th>Solution Times</th>
<th>Counting and</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Retrieval Trials</td>
<td>Fingers Trials(^c)</td>
<td>Counting-Fingers Trials</td>
<td>Retrieval Trials</td>
</tr>
<tr>
<td>Group 1</td>
<td>.67</td>
<td>.42</td>
<td>.75</td>
<td>.10</td>
</tr>
<tr>
<td>Group 2</td>
<td>.56</td>
<td>.66</td>
<td>.50</td>
<td>.00</td>
</tr>
<tr>
<td>Group 3</td>
<td>.70</td>
<td>.26</td>
<td>.74</td>
<td>.38</td>
</tr>
</tbody>
</table>

\(^a\)Numbers in the Table indicate correlations of percentage of overt strategy use on each problem with the variable specified in the Table. For example, the top left value of .83 indicates a raw correlation of \( r = .83 \) between percentage of errors on retrieval trials on each problem and percentage of overt strategy use on that problem for Group 1.

\(^b\)Summing across the performance of children in all three groups, the correlation was \( r = .68 \).

\(^c\)Summing across the performance of children in all three groups, the correlation was \( r = .66 \).

tween the correlations for Group 1 was \( t(22) = 2.82 \); for Group 2, \( t(22) = 2.26 \); and for Group 3, \( t(22) = 5.03 \) (all \( p \)'s \(< .05 \)). For the partial correlations, the difference between the correlations' magnitudes was significant for Group 3, \( t(22) = 2.81, p < .01 \). It did not reach significance for Group 1, \( t(22) = 1.64 \), or for Group 2, \( t(22) = 1.32 \), despite the rather large differences between the correlations (\( r = .67 \) versus \( r = .42 \) for Group 1 and \( r = .56 \) versus \( r = .23 \) for Group 2).

The model predicted that the correlation between the percentage of errors on fingers-strategy trials and percentage of overt strategy use would be intermediate between the correlations of overt strategy use with errors on retrieval trials and with errors on counting and counting-fingers trials. Children made too few errors on fingers trials on each problem for us to analyze their performance separately in each of the three groups. Children in Group 1, for example, erred only twice on fingers trials. Combining across the three groups, however, the correlation between percentage of errors on fingers-strategy trials on each problem and
percentage of overt strategy use on that problem was $r = .68$, which was, as predicted, intermediate between the other two correlations. With the contribution of the sum partialed out, the correlation was $r = .66$.

The source of the correlation between solution times and overt strategy use was also in accord with the prediction of the model. In all three groups, the best predictor of mean solution times on retrieval trials on each problem was percentage of overt strategy use on that problem (Appendix A). In contrast, in all three groups the sum was a better predictor of mean solution times on counting and counting-fingers trials on each problem than was the percentage of overt strategy use on that problem. The differences in the magnitudes of the raw correlations was significant for Group 1, $t(22) = 2.60, p < .05$, though not for the other two groups, $t's(22) = 1.45$ and $84$. The comparisons of the partial correlations yielded more striking results. As shown in Table 9.3, large differences separated the magnitudes of the partial correlations involving solution times on retrieval and on counting and counting-fingers trials. The differences were significant for all three groups ($t's(22) = 3.38, 2.24,$ and $5.07$ for Groups 1, 2, and 3, respectively, all $p's < .05$).

Explicit Judgment Data. Explicit judgments of problem difficulty collected from the same subjects who provided the addition performance data showed a similar pattern to that previously reported with between-subjects data. The correlations between mean rating of problem difficulty for each problem and percentages of errors on that problem were $r = .64$, $r = .34$, and $r = .70$ for Groups 1, 2, and 3, respectively. The correlations between mean rating of difficulty for each problem and percentage of overt strategy use on that problem were $r = .73$, $r = .50$, and $r = .53$, respectively. All six of these correlations were lower than any of the three correlations between percentages of overt strategy use and errors. Four of the six were also lower than the corresponding correlations between the metacognitive judgments and the size of the larger addend, suggesting that it was not unreliability of measurement of the metacognitive judgments that led to the lower correlations. The slippage that would be entailed in going from problem difficulty to judgments of problem difficulty and then from judgments of problem difficulty to use of overt strategies added to the unlikelihood that the accuracy of the metacognitive judgments of difficulty could account for the correlations between overt strategy use and errors.

Summary of Replication and Extension Experiment Findings. The results of the replication and extension experiment indicated that the earlier results were not due to any peculiarity of the problem set. The data from each of the three groups closely paralleled the Siegler and Robinson findings concerning the existence of the four strategies, their relative accuracy, their relative solution times, the correlations among percentage of overt strategy use, mean solution time, and percentage of errors on each problem, and the source of these correlations being
errors and solution times on retrieval trials. On the new problems, as on the
original ones, young children demonstrated the ability to make adaptive strategy
choices even without the ability to make highly accurate judgments about prob-
lem difficulty.

AN EXPERIMENT ON SUBTRACTION

Although the model was developed to account for strategy choices on addition
problems, it seemed to provide a plausible model of strategy choices on simple
subtraction problems as well. To use the counting-fingers strategy, children
could raise fingers to represent the larger number, lower fingers corresponding to
the smaller number, and count the remaining fingers. To use the fingers strategy,
they could put up fingers representing the larger number, put down fingers
representing the smaller number, and answer without counting the remainder. To
use the counting strategy, they could count backward from the larger number, or
count up to it from one and then count backward, without putting up fingers. To
use the retrieval strategy, they could retrieve an answer and state it without any
intervening visible or audible behavior. The expectations for the relative solution
times of the strategies, the relations among percentage of overt strategy use,
mean solution times, and percentage of errors on each problem, and the source of
these relations' being errors and solution times on retrieval trials would be the
same as in addition. The subtraction experiment was performed to test whether
subtraction did resemble addition in these ways.

Method

Participants were 34 children, half 5-year-olds and half 6-year-olds, half of each
age group boys and half girls. The 5-year-olds attended a university preschool.
The 6-year-olds attended the first grade of an upper-middle-class suburban
school.

The problems were the inverses of the 25 addition problems presented by
Siegler and Robinson. For every problem of the form $a + b = c$ in the Siegler
and Robinson study, here there was a problem of the form $c - b = a$. The
instructions were:

I want you to imagine that you have a pile of oranges. Then imagine that I take
some oranges away from your pile. Tell me how many you have left. You can do
anything you want to help you get the right answer. If you want to use your fingers
or count aloud, that's fine. Okay? Suppose you have $m$ oranges, how many would
you have if I took away $n$ of them?